CRITICAL BUCKLING LOAD SOLUTION OF THIN BEAM ON WINKLER FOUNDATION VIA POLYNOMIAL SHAPE FUNCTION IN STODOLA-VIANELLO ITERATION METHOD

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Abstract
The critical buckling load analysis of thin beam on Winkler foundation (BoWF) is crucial for BoWF subjected to in-plane compressive load. This paper presents the Stodola-Vianello method for the approximate solution of the governing ordinary differential equation (ODE). The Stodola-Vianello method expresses the ODE in iteration form after four successive integrations have been used in reformulating the equation. The problem then reduces to iteration in algebra. By deriving algebraic buckling shape functions that satisfy the boundary conditions, and substitution into the Stodola-Vianello iteration formula derived successive iterations of the buckling shape function would yield better approximations of the buckling shape. The existence condition for convergence which is the identity of the *n*th and $(n+1)$ iterations are used to find the eigenvalue from which the buckling load is determined. The candidate problem solved is a BoWF with Dirichlet boundary conditions. It is found that one iteration yields sufficiently accurate buckling loads which differs from the exact solution by 0.129% for $\beta l^4 = 0$, 0.0807% for $\beta l^4 = 50$, and 0.0567% for $\beta l^4 = 100$, where βl^4 is a parameter measuring the beam structure – Winkler foundation interaction. The negligible difference in the critical buckling load found is because an approximate algebraic one-parameter shape function was used in the iteration. The paper has demonstrated the effectiveness of the Stodola-Vianello iteration for solving the stability of thin beam on Winkler foundation.

Key Words - Stodola-Vianello iteration method, algebraic buckling shape function, convergence, critical buckling load, beam on Winkler foundation parameter

1. Introduction

Beam on Winkler foundation (BoWF) problems where the beam is subjected to compressive forces are commonly encountered in structural and geotechnical engineering. Such structures are prone to buckling failures and it is vital to determine the least compressive loads that could cause buckling failure.

The two most common beam theories are Euler-Bernoulli beam theory (EBBT) and Timoshenko beam theory (TBT). Other beam theories have however been formulated to consider shear deformation by Dahake and Ghugal [1], Levinson [2], Sayyad and Ghugal [3] and many others.

Elastic foundation models have also been formulated based on continuum assumptions and discrete parameter assumptions of the soil. Discrete parameter foundation models have been variously proposed by Winkler, Vlasov, Pasternak, Hetenyi, Kerr and other researchers.

The classical thin beam theory (CTBT) also called EBBT assumes the Bernoulli-Navier hypothesis that cross-sectional planes that are normal to the longitudinal axis remain plane and normal to the longitudinal axis even after deformation. This orthogonality assumption implies that transverse shear deformation is ignored in the formulation of the beam equations, thus rendering the theory ideal for thin beams where the contributions of shear deformation can be ignored [4].

For moderately thick and thick beams, EBBT has been found to yield significant errors, and this had prompted the development of theories and models that consider shear deformation [5]. Such shear deformation beam theories have been developed and studied by Timoshenko, Dahake and Ghugal [1], Levinson [2], Sayyad and Ghugal [3], Ike [5].

This paper considers thin beams and hence EBBT is used.

Winkler model considers the soil as closely spaced, independent linear elastic springs with stiffness that is directly proportional to the deflection. The soil reaction is thus proportional to the deflection of the beam at the considered point. The Winkler soil parameter, which indicates the soil spring constant/stiffness is the one-parameter model used to describe the soil reaction. The major defect of the Winkler idealization is the assumed lack of continuity of the soil spring model occasioned by the mutually independent soil springs used in the model. Thus shear interaction affects are ignored in the Winkler model.

Other discrete parameter foundation models that account for shear interaction of the discrete springs were proposed by Pasternak, Hetenyi, Vlasov and others. Two-parameter elastic discrete foundation models use two-parameter to describe the soil reaction, the first parameter represents the vertical stiffness while the second parameter represents the coupling interaction effect of the vertical springs.

This paper models the foundation using the Winkler model.

Hetenyi [6] used the classical methods of solving boundary value problems to derive exact critical buckling load solutions of beams on Winkler foundations. Timoshenko and Gere [7] and Wang et al [8] have studied BoWF and developed exact critical buckling load expressions for BoWF under Dirichlet conditions. Taha and Hadima [9] and Taha [10] studied BoWF using recursive differentiation method (RDM) and obtained analytical expressions for least buckling loads for non-prismatic BoWF.

Atay and Coskun [11] studied the BoWF problems using variational iteration methods (VIMs). They obtained critical buckling load solutions for BoWF for prismatic and non-prismatic cross-sectional properties. Anghel and Mares [12] presented collocation method based on integral formulation for the buckling analysis of BoWF. Hassan [13] investigated the stability of BoWF for various end supports. Aristizabal-Ochoa [14] studied the stability of BoWF for various boundary conditions. Soltani [15] studied the stability of BoWF using finite element techniques.

Ike [16] studied BoWF under dynamic excitation using finite sine transform method and derived exact eigen solutions. Ofondu et al [17] applied the Stodola-Vianello iteration formulation of the stability problem of Euler columns to solve for critical buckling loads. They considered Euler column buckling problem with clamped-pinned ends and used an algebraic basis function that was constructed to satisfy the boundary conditions to derive successive iterates for the basis buckling function. They found that a few iterative steps gave accurate critical buckling load solutions.

Ike [18] used a point collocation method to approximately solve BoWF problems by requiring that the governing equation be exactly satisfied only at discrete points on the domain, rather that at all points on the domain. He obtained satisfactory approximations. Ike et al [19] have studied stability of Euler columns using Picard's iterative method and obtained accurate buckling load results for pinned ends. Ikwueze et al [20] have used least squares weighted residual methods for buckling solutions of Euler columns with fixed-pinned ends.

Mama et al [21] used quintic polynomial shape functions in a finite element formulation to obtain accurate buckling load solutions to BoWF problems. Literature shows that the Stodola-Vianello method has not been applied to BoWF buckling analysis.

In this paper, the Stodola-Vianello iterative method is used to derive buckling load solutions for BoWF with Dirichlet boundary conditions.

2. Theory

The equation for the stability of thin beam on Winkler foundation (BoWF) is:

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2v(x)}{dx^2}\right)+P\frac{d^2v(x)}{dx^2}+kv(x)=p(x) \qquad (1)$$

where $v(x)$ is the transverse deflection, x is the longitudinal coordinate axis, k is the Winkler modulus, P is the compressive load, $p(x)$ is the distributed transverse load, E is the Young's modulus, I is the moment of inertia.

When there is no transversely applied load, $p(x)=0$ and for homogeneous prismatic beams, the governing equation is the fourth order ordinary differential equation given by:

$$\frac{d^4v(x)}{dx^4}+\alpha\frac{d^2v(x)}{dx^2}+\beta v(x)=0 \qquad (2)$$

where $\alpha=\frac{P}{EI}$

(3)

$$\beta=\frac{k}{EI} \qquad (4)$$

3. Methodology

By re-arrangement of Equation (2) and four successive integrations, the Stodola-Vianello iteration equations are found.

$$\frac{d^4v}{dx^4}=-\left(\alpha\frac{d^2v}{dx^2}+\beta v\right) \qquad (5)$$

Integrating Equation (5) with respect to x gives:

$$\frac{d^3v(x)}{dx^3}=-\alpha\frac{d^2v}{dx}-\beta\int_0^x v(x)\,dx+c_1 \qquad (6)$$

where c_1 is an integration constant.

Integrating Equation (6) with respect to x gives:

$$\frac{d^2v}{dx^2}=-\alpha v(x)-\beta\int_0^x\int_0^x v(x)\,dx\,dx+c_1x+c_2 \qquad (7)$$

c_2 is the second integration constant.

Integrating Equation (7) with respect to x gives:

$$\frac{dv(x)}{dx}=\theta(x)=-\alpha\int_0^x v(x)\,dx-\beta\int_0^x\int_0^x\int_0^x v(x)\,dx\,dx\,dx+\frac{c_1x^2}{2}+c_2x+c_3 \qquad (8)$$

where c_3 is the third integration constant, and $\theta(x)$ is the slope.

Integrating Equation (8) with respect to x gives:

$$v(x) = -\alpha\int_0^x\int_0^x v(x)\,dx\,dx - \beta\int_0^x\int_0^x\int_0^x\int_0^x v(x)\,dx\,dx\,dx\,dx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad \ldots(9)$$

where c_4 is the fourth integration constant.

4. Results

The stability of the pinned-pinned beam on Winkler foundation shown in Figure 1 is considered.

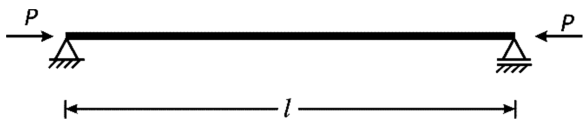

Figure 1: Stability problem of pinned end beam on Winkler foundation

An algebraic polynomial shape function is derived from the fourth degree polynomial:

$$v(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \quad (10)$$

where a_0, a_1, a_2, a_3 and a_4 are the polynomial constants.
$v(x)$ is required to satisfy the simply supported conditions of the beam at the left and right supports. Thus,

$$v(0) = v''(0) = 0 \quad (11a)$$

$$v(l) = v''(l) = 0 \quad (11b)$$

Thus

$$v''(x) = 2a_2 + 6a_3x + 12a_4x^2 \quad (12)$$

$$v(0) = a_0 = 0 \quad (13)$$

$$v''(0) = 2a_2 = 0 \quad (14)$$

$$\therefore \; a_2 = 0 \quad (14a)$$

$$v''(l) = 6a_3l + 12a_4l^2 = 0 \quad (15)$$

$$a_3 = -2a_4l \quad (15a)$$

$$v(l) = a_1l + a_3l^2 + a_4l^4 = 0 \quad (16)$$

$$a_1 = a_4l^3 \quad (16a)$$

Hence,

$$v(x) = a_4l^3x - 2a_4lx^3 + a_4x^4 = a_4(x^4 - 2lx^3 + l^3x) \quad (17)$$

The Stodola-Vianello equations become:

$$v''(x) = -\alpha a_0(x^4 - 2lx^3 + l^3x) - \beta a_0\int_0^x\int_0^x (x^4 - 2lx^3 + l^3x)dx\,dx + c_1x + c_2 \quad \ldots(18)$$

Simplifying,

$$v''(x) = -\alpha a_0(x^4 - 2lx^3 + l^3a) - \beta a_0\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) + c_1x + c_2 \quad (19)$$

Using the boundary condition:

$$v''(0) = c_2 = 0 \quad (20)$$

$$v''(l) = -\beta a_0(0.1l^6) + c_1l = 0 \quad (21)$$

$$\therefore \quad c_1 = 0.1a_0l^5\beta \quad (22)$$

Also,

$$v(x) = -\alpha a_0\int_0^x\int_0^x (x^4 - 2lx^3 + l^3x)dx\,dx - \beta a_0\int_0^x\int_0^x\int_0^x\int_0^x (x^4 - 2lx^3 + l^3x)dxdxdxdx + \frac{0.1a_0l^5x^3\beta}{6} + c_3x + c_4 \quad (23)$$

Integrating,

$$v_1(x) = -\alpha a_0\left(\frac{x^6}{6} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta a_0\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{0.1a_0l^5x^3\beta}{6} + c_3x + c_4 \quad \ldots(24)$$

$$v(0) = c_4 = 0 \quad (25)$$

$$v(l) = -\alpha a_0(0.1l^6) + 0.010119047a_0l^8\beta + c_3l = 0 \quad (26)$$

$$c_3 = 0.1a_0l^5\alpha - 0.010119047a_0l^7\beta \quad (27)$$

Hence,

$$v_{n+1}(x) = -\alpha a_0\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta a_0\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{a_0l^5x^3\beta}{60} + (0.1a_0l^5\alpha - 0.010119047a_0l^7\beta)x \quad (28)$$

At convergence,

$$v_{n+1}(x) = v_n(x) \quad (29)$$

So,

$$a_0(x^4 - 2lx^3 + l^3x) = -a_0\alpha\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta a_0\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{a_0l^5x^3\beta}{60} + 0.1a_0l^5x\alpha - 0.010119047a_0l^7x\beta \quad (30)$$

Integrating both sides,

$$\int_0^l v_{n+1}(x)\,dx = \int_0^l v_n(x)\,dx \quad (31)$$

$$\int_0^l a_0(x^4 - 2lx^3 + l^3x)\,dx = \int_0^l \left\{-a_0\alpha\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta a_0\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{a_0l^5x^3\beta}{60} + 0.1a_0l^5x\alpha - 0.010119047a_0l^7x\beta\right\}dx \quad \ldots(32)$$

Evaluating the integral and rearranging,

$$a_0\left[\frac{x^5}{5} - \frac{2lx^4}{4} + \frac{l^3x^2}{2}\right]_0^l = a_0\left[-\alpha\left(\frac{x^7}{210} - \frac{lx^6}{60} + \frac{l^3x^4}{24}\right) -\right.$$

$$\beta\left(\frac{x^9}{15120}-\frac{lx^8}{3360}+\frac{l^3x^6}{720}\right)+$$

$$\left.\frac{l^5x^4}{240}\beta+\frac{l^5x^2}{20}-5.0595235\times10^{-3}l^7x^2\beta\right]_0^l \qquad (33)$$

Substituting the limits and simplifying gives:

$$\frac{l^5}{5}+2.050264241\times10^{-3}\beta l^9 = 0.020238095\alpha l^7 \qquad (34)$$

Solving for α,

$$\alpha=\frac{P}{EI}=9.882352941l^{-2}+0.101307174\beta l^2 \qquad (35)$$

Hence,

$$P_{cr}=\frac{EI}{l^2}(9.88235941+0.101307174\beta l^4) \qquad (36)$$

$$P_{cr}=\frac{EI}{l^2}K(\beta l^4) \qquad (37)$$

$$K(\beta l^4)=9.882352941+0.101307174\beta l^4 \qquad (38)$$

$K(\beta l^4)$ is the critical buckling load coefficient.

$K(\beta l^4)$ is determined for various values of βl^4 and tabulated in Table 1.

Table 1
Critical buckling load coefficients of various values of βl^4 and comparison with exact results

βl^4	Present study	Exact Wang et al [8](2005)	Error %
0	9.882352941	9.869604401	0.129
50	14.94771165	14.93566358	0.0807
100	20.01307034	20.00172277	0.0567

5. Discussion

This study has presented critical buckling load analysis of Euler-Bernoulli beam on Winkler foundation using polynomial shape function in the Stodola-Vianello iteration method.

A one-parameter polynomial shape function was derived for simply supported thin beam on Winkler foundations in Equation (17) such that the boundary conditions are satisfied. The function is then used in the Stodola-Vianello iteration formula to generate the next iteration for the buckled deflection shape function after the integration constants are found via enforcement of boundary conditions. The next buckled deflection function is found as Equation (28).

The condition of convergence given by Equations (29) and (30) is then applied to find the characteristic buckling equation as Equation (35). The critical buckling load is then found as Equation (36) and in terms of the critical buckling load coefficient $K(\beta l^4)$ as Equation (37). $K(\beta l^4)$ is presented in Table 1 for various values of βl^4 and illustrates the accuracy of the results obtained using algebraic one-parameter polynomial function in the Stodola-Vianello iteration method.

6. Conclusion

In conclusion

(i) Stodola-Vianello iteration method for BoWF problems simplifies the solution to solving iteration problems in algebra.

(ii) The first approximation to the buckling shape function is derived to satisfy the boundary conditions of the BoWF.

(iii) For the BoWF with pinned-pinned ends one iteration produces buckling load solutions with errors of 0.129% for $\beta l^4=0$, 0.0807% for $\beta l^4=50$ and 0.0567% for $\beta l^4=100$.

(iv) The errors are due to the approximate nature of the algebraic buckling shape function used in the Stodola-Vianello iteration, and expectedly the resulting successive iteration can only yield approximate result.

Notations and Abbreviations

ODE(s)	Ordinary Differential Equation(s)
BoWF	Beam on Winkler Foundation
EBBT	Euler-Bernoulli Beam Theory
TBT	Timoshenko Beam Theory
CTBT	Classical Thin Beam Theory
RDM	Recursive Differentiation Method
VIM	Variational Iteration Method
x	longitudinal coordinate axis
$v(x)$	transverse deflection or buckled deflection function
k	Winkler modulus
P	compressive load
$p(x)$	distributed transverse load
E	Young's modulus
I	moment of inertia
α	beam buckling parameter that depends on compressive load (P) and the beam flexural modulus (EI)
β	beam on Winkler foundation parameter that depends on the Winkler parameter k and the beam flexural modulus (EI)
c_1, c_2, c_3, c_4	integration constants
a_0, a_1, a_2, a_3, a_4	constants of the algebraic polynomial function used to define the buckled shape function $v(x)$
$\theta(x)$	slope of beam
P_{cr}	critical buckling load
$K(\beta l)$	critical buckling load coefficient in dimensionless form
$\int \ldots dx$	integral
$\iint \ldots dx$	double integral (successive integral)
$v'(x)$	derivative of $v(x)$ with respect to x
$v''(x)$	second derivative of $v(x)$ with respect to x

$\frac{d}{dx}$	ordinary differential operator, ordinary derivative with respect to x
$\frac{d^n}{dx^n}$	nth ordinary differential operator
$v_n(x)$	nth buckled shape of beam
$v_{n+1}(x)$	$(n + 1)$th iterate of $v_n(x)$

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