KANTOROVICH VARIATIONAL METHOD FOR THE ELASTIC BUCKLING ANALYSIS OF KIRCHHOFF PLATES WITH TWO OPPOSITE SIMPLY SUPPORTED EDGES

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Abstract

We present the Kantorovich variational method for the elastic buckling solutions of Kirchhoff plates with two opposite simply supported edges. Uniform compressive loading is applied on the two simply supported edges, $x = 0$ and $x = a$. Three cases of edge support considered along the $y = 0$ and $y = b$ edges are (i) both are clamped (ii) both are simply supported, and (iii) edge $y = 0$ is simply supported while edge $y = b$ is free. The Kantorovich method which is based on seeking to minimize the total potential energy functional Π with respect to the unknown deflection $w(x, y)$ assumes the deflection in variable separable form as an infinite series of the product of an unknown function $f_m(y)$ and a known sinusoidal function that satisfies the Dirichlet boundary conditions along the simply supported edges. Euler-Lagrange differential equation is used to obtain the fourth order ordinary differential equation which $f_m(y)$ must satisfy to minimize Π. The general solution for $f_m(y)$ is obtained as a combination of hyperbolic and trigonometric functions with four unknown integration constants. The conditions for nontrivial solutions are applied to find the characteristic stability equations in each considered case. Exact solutions are obtained for the elastic buckling equations which are identical with previously obtained solutions in the literature. The elastic buckling equations are solved by iteration methods and by closed form solution methods for the Kirchhoff plate to obtain the elastic buckling loads expressed in terms of the elastic buckling load coefficients. It was found that the method yields exact elastic buckling load and exact elastic buckling load coefficient for each boundary condition considered in this study.

Key Words - Kantorovich variational method, Kirchhoff plate, Total potential energy functional, Euler – Lagrange differential equation, Characteristic elastic buckling equation

1. Introduction

Elastic buckling problems of plates occur for plates subject to in-plane compressive loads which may be uniformly or non-uniformly distributed [1–7].

Navier presented a derivation of the governing partial differential equation PDE for the problem of isotropic, homogeneous, thin rectangular plates. His derivation which laid the groundwork for much of thin plate elastic buckling research used the Kirchhoff's assumptions and included the twisting term which represented a significant contribution to the knowledge and theory of elastic stability at the time.

Saint Venant formulated an improvement on the Navier's governing domain PDE of elastic buckling of isotropic homogeneous thin rectangular plates.

Since the work of Navier and Saint Venant several other ground breaking research work on the theme of elastic and in-elastic buckling of plates of various shapes, sizes subjected to different types of loads and for different boundary conditions have been reported.

Some research work that reported significant contributions to the elastic and inelastic buckling research include: Timoshenko [1], Timoshenko and Gere [2], Bulson [3], Chajes [4], Gambhir [5], Wang et al [6], Shi and Bezine [7], Shi [8], Abodi [9], Batford and Houbolt [10], Wang et al [11], Ullah et al [12, 13], Xiang et al [14], Yu Chen [15] and Abolghasemi et al [16].

Ibearugbulem [17] used the Ritz direct variational method to determine buckling loads of isotropic, homogeneous, thin flat plates under to in-plane uniform compressive loading for various boundary conditions. Though one parameter shape functions were used he obtained solutions that were not significantly different from the previous results from literature.

Nwadike [18] also used the Ritz method to find buckling loads of isotropic, homogeneous thin flat plates under in-plane uniaxial uniform compressive forces for various boundary conditions. Using one parameter shape functions, he obtained reasonably good results that compared well with solutions from previous literature.

Oguaghamba [19] investigated the buckling and post buckling loads of isotropic, homogeneous thin rectangular flat plates. Oguaghamba et al [20] studied the buckling and post buckling load characteristics of isotropic, homogenous thin rectangular flat plates with all edges clamped.

Abodi [9] presented a finite difference method for solving the elastic buckling problem of simply supported thin rectangular plates under in-plane patch loads, and obtained the elastic buckling loads, by solving the resulting finite difference equations.

Shi and Bezine [7] and Shi [8] solved the stability problems of orthotropic plates with the aid of boundary element method.

Ibearugbulem et al [21] and Osadebe et al [22] used the Taylor – Maclaurin's series to formulate displacement shape functions for simply supported Euler – Bernoulli beams and applied such formulated shape functions in the Galerkin's method to find the elastic buckling loads of simply supported isotropic, homogeneous thin rectangular plates under in-plane uniaxial, uniform compressive loads.

Nwoji et al [23] applied two dimensional finite Fourier sine integral transform method to determine the elastic buckling loads of thin rectangular plates under uniaxial and biaxial uniform compressive loads for the case of Dirichlet boundary conditions along all the four edges. For the case of uniaxial compressive load, they considered the load applied along the two opposite edges $x = 0$ and $x = a$, (where a corner point is considered the origin) while for biaxial uniform compressive load, the loads were applied along $x = 0$, $x = a$, $y = 0$ and $y = b$. They found the integral kernel function of the double finite Fourier sine transformation ideally suited to the problem since the Dirichlet boundary conditions were identically satisfied along all the boundaries. They found that the double integral transform converts the elastic buckling problem represented in equilibrium form as a PDE to an integral equation, and ultimately to an algebraic problem that gives closed form expressions for stability problem for all the buckling modes.

Onah et al [24] used one-dimensional finite Fourier sine integral transform method to obtain the elastic buckling loads of thin rectangular plates with two opposite edges ($x = 0$ and $x = a$) simply supported and the other edges ($y = 0$ and $y = b$) clamped with uniform compressive force applied at the simply supported edges. They found the one-dimensional finite Fourier sine integral transform used ideally fitted to the problem since the sinusoidal kernel function used satisfied the Dirichlet boundary conditions.

Finite Fourier sine integral transform with respect to x coordinate variable converted the governing domain PDE to an integral equation over the plate domain, which further simplified to an ordinary differential equation (ODE). They solved the ODE subject to the clamped edge conditions obtaining the characteristic elastic buckling equations using the requirements for non-trivial solutions on the resulting homogeneous set of equations. They solved the characteristic buckling equation which is a transcendental equation using computer based iteration tools to obtain exact solutions for the buckling loads that agreed with previous solutions in the literature.

Iyengar [25] solved the problem of buckling of thin plates for various support conditions, and obtained exact solutions. for the case of Dirichlet boundary conditions.

Onyia et al [26] applied Galerkin-Kantorovich method for solving stability problems of thin SCSC plates. Onyia et al [27] used the one-dimensional finite Fourier sine integral transform method to obtain exact solutions to the elastic stability problems of rectangular thin SSSS and SSCF plates. Onyia et al [28] has used the Galerkin-Vlasov variational method for solving stability problems of rectangular thin SSCF and SSSS plates, and obtained exact solutions.

Oguaghamba and Ike [29] used the Galerkin-Vlasov method for the analysis of the stability problems of thin plates with three simply supported edges and one free edge for the case of uniform uniaxial compression.

In this work, the Kantorovich variational method is presented for solving stability problems of Kirchhoff plates with two opposite simply supported edges subjected to uniform in-plane compressive load; while the other two edges are: clamped (SCSC or CSCS plate) or simply supported (SSSS plate) or one edge is simply supported and the other edge is free (SSSF plate). Hence Kantorovich variational method is presented in this work for solving the elastic buckling problems of SCSC (or CSCS). Kirchhoff plates, SSSS Kirchhoff plates and SSSF Kirchhoff plates.

1.1. Justification for the study

Review of literature shows that the variational Kantorovich method had not been previously applied to the solution of the elastic buckling problems of Kirchhoff plates with two opposite simply supported edges. The study is thus novel as it demonstrates the effectiveness of the variational Kantorovich methodology as a potent mathematical tool for solving the elastic buckling problem of thin plates posed in variational form.

2. Theoretical Framework

The total potential energy functional Π for a rectangular Kirchhoff plate subjected to in-plane edge compressive forces N_x, N_y and shear force N_{xy} is:

$$\Pi = U + V \tag{1}$$

in which U is the strain energy, and V is the potential energy of the loading.

For isotropic homogeneous thin rectangular plates, U is expressed as:

$$U = \frac{1}{2} D \iint_A \left\{ (\nabla^2 w)^2 - 2(1-\mu)\left(w_{xx} w_{yy} - (w_{xy})^2 \right) \right\} dxdy \qquad (2)$$

wherein $w(x, y)$ denotes deflection and A represents the plate domain, μ denotes the Poisson's ratio of the plate material, D is the plate flexural rigidity given by

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad (3)$$

wherein E denotes the Young's modulus of elasticity, h is the thickness of the plate.

The potential function of the external forces is:

$$V = -\frac{1}{2} \iint \left\{ N_{xx} \left(\frac{\partial w}{\partial x} \right)^2 + N_{yy} \left(\frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} dxdy \qquad \ldots(4)$$

Hence,

$$\Pi = \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\mu) \left(\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) \right\} dxdy - \frac{1}{2} \iint \left\{ N_{xx} \left(\frac{\partial w}{\partial x} \right)^2 + N_{yy} \left(\frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} dxdy \qquad \ldots(5)$$

Simplifying,

$$\Pi = \frac{D}{2} \iint \left\{ (\nabla^2 w)^2 + 2(1-\mu)(w_{xy}^2 - w_{xx} w_{yy} \right\} dxdy - \frac{1}{2} \iint \left\{ N_{xx} (w_x)^2 + N_{yy} (w_y)^2 + 2N_{xy} w_x w_y \right\} dxdy \qquad (6)$$

where ∇^2 is the Laplacian expressed by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$w_{xy} = \frac{\partial^2 w}{\partial x \partial y}$$

$$w_x = \frac{\partial w}{\partial x}$$

$$w_y = \frac{\partial w}{\partial y}$$

$$w_{xx} = \frac{\partial^2 w}{\partial x^2}$$

$$w_{yy} = \frac{\partial^2 w}{\partial y^2}$$

The elastic buckling problem considered in this work is shown in Figure 1.

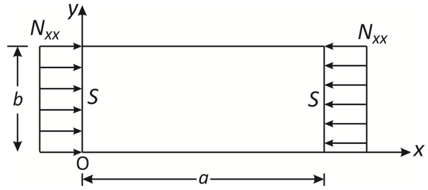

Figure 1: Rectangular thin plate with simply supported boundaries ($x = 0$, $x = a$) under uniform compressive load while edges are subject to various boundary conditions

The problem considered shown in the Figure 1 is a thin plate having the two opposite sides $x = 0$ and $x = a$ simply supported, and the other sides $y = 0$, and $y = b$ subject to three cases of boundary conditions namely: (i) both are clamped, (ii) both are simply supported and (iii) edge $y = 0$ is simply supported and edge $y = b$ is free. The compression force is applied only along the simply supported sides while the other sides are not subject to any loading. Thus $N_{yy} = 0$, and $N_{xy} = 0$.

The energy functional then simplifies to:

$$\Pi = \frac{D}{2} \iint_A \left\{ (\nabla^2 w)^2 - 2(1-\mu)(w_{xx} w_{yy} - w_{xy}^2) \right\} dxdy - \frac{1}{2} \iint_A N_{xx} (w_x)^2 dxdy \qquad (7)$$

The objective of stability problems of thin plates is to find minimum total potential energy functional with respect to $w(x, y)$.

Three cases considered are (i) the sides $y = 0$ and $y = b$ are clamped (Figure 2) in which case the clamped boundary conditions are:

$$\begin{aligned} w(x, y=0) &= w(x, y=b) = 0 \\ w_y(x, y=0) &= w_y(x, y=b) = 0 \end{aligned} \qquad (8)$$

and (ii) the sides $y = 0$ and $y = b$ are simply supported (Figure 3) with corresponding boundary conditions

$$\begin{aligned} w(x, y=0) &= w(x, y=b) = 0 \\ w_{yy}(x, y=0) &= w_{yy}(x, y=b) = 0 \end{aligned} \qquad (9)$$

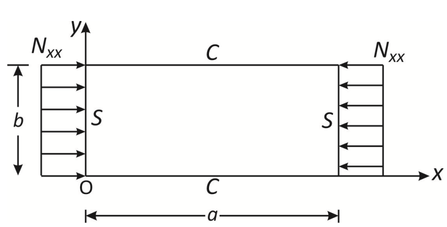

Figure 2: CSCS Kirchhoff plate under uniform uniaxial compression N_{xx} on the simply supported sides

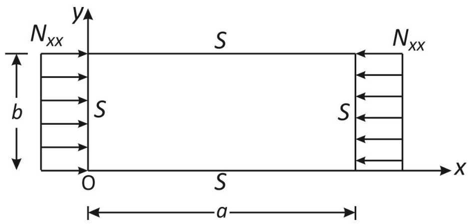

Figure 3: SSSS Kirchhoff plate under uniform uniaxial compression N_{xx} on the simply supported sides

(iii) Three sides $x = 0$, $x = a$, and $y = 0$ are on simple supports while the side $y = b$ is free as shown in Figure 4.

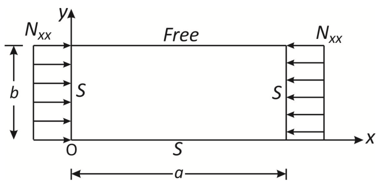

Figure 4: SSSF Kirchhoff plate simply supported along three edges $x = 0$, $x = a$ and $y = 0$, and free along $y = b$

The SSSF Kirchhoff plate is under uniform uniaxial compression along the simply supported sides shown in Figure 4.

The boundary conditions are:

$$w(x=0, y) = w(x=a, y) = 0$$

$$\frac{\partial^2 w}{\partial x^2}(x=0, y) = \frac{\partial^2 w}{\partial x^2}(x=a, y) = 0 \tag{10}$$

$$w(x, y=0) = 0$$

$$M_{yy}(x, y=0) = -D(w_{yy} + \mu w_{xx})\big|_{x, y=0} = 0$$

$$M_{yy}(x, y=b) = -D(w_{yy} + \mu w_{xx})\big|_{x, y=b} = 0$$

$$V_y(x, y=b) = -D(w_{yyy} + (2-\mu)w_{xxy})\big|_{x, y=b} = 0$$

where $w_{xxy} = \dfrac{\partial^3 w}{\partial x^2 \partial y}$, $w_{yyy} = \dfrac{\partial^3 w}{\partial y^3}$

M_{yy} is bending moment, V_y is the effective shear force.

3. Methodology

3.1. Deflection function

The expression for buckled deflection is chosen in variable separable form as the infinite series:

$$w(x, y) = \sum_{m=1}^{\infty} f_m(y) g_m(x) \tag{11}$$

$m = 1, 2, 3, \ldots, \infty$

where $g_m(x)$ is chosen to apriori satisfy Dirichlet conditions for $x = 0$, and $x = a$ and $f_m((y)$ is the function sought such that Π is minimized. Thus $w(x, y)$ is considered as the infinite series expression.

$$w(x, y) = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a} \tag{12}$$

The problem then reduces to finding the unknown function $f_m(y)$ that extremizes the total potential energy functional Π and also satisfies the boundary conditions along the edges $y = 0$ and $y = b$.

3.2. Total potential energy functional

By differentiation of $w(x, y)$, we obtain:

$$w_x = \sum_{m=1}^{\infty} \frac{m\pi}{a} f_m(y) \cos \frac{m\pi x}{a} \tag{13}$$

$$w_{xx} = \sum_{m=1}^{\infty} -\left(\frac{m\pi}{a}\right)^2 f_m(y) \sin \frac{m\pi x}{a} \tag{14}$$

$$w_{xy} = \sum_{m=1}^{\infty} \left(\frac{m\pi}{a}\right) f_m'(y) \cos \frac{m\pi x}{a} \tag{15}$$

where $f_m'(y) = \dfrac{df_m(y)}{dy}$

$$w_y = \sum_{m=1}^{\infty} f_m'(y) \sin \frac{m\pi x}{a} = \sum_{m=1}^{\infty} \frac{df_m(y)}{dy} \sin \frac{m\pi x}{a} \tag{16}$$

$$w_{yy} = \sum_{m=1}^{\infty} \frac{d^2 f_m(y)}{dy^2} \sin \frac{m\pi x}{a} = \sum_{m=1}^{\infty} f_m''(y) \sin \frac{m\pi x}{a} \tag{17}$$

$$\nabla^2 w = w_{xx} + w_{yy} = \sum_{m=1}^{\infty} \left(f_m''(y) - \left(\frac{m\pi}{a}\right)^2 f_m(y) \right) \sin \frac{m\pi x}{a} \quad \ldots(18)$$

$$(\nabla^2 w)^2 = \left\{ \sum_{m=1}^{\infty} \left(f_m''(y) - \left(\frac{m\pi}{a}\right)^2 f_m(y) \right) \sin \frac{m\pi x}{a} \right\}^2 \tag{19}$$

Then,

$$\Pi = \frac{D}{2} \iint \sum\sum \left\{ (f_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) f_m(y) + \left(\frac{m\pi}{a}\right)^4 f_m^2(y) \right\} \sin^2 \frac{m\pi x}{a} - 2(1-\mu) \times \left\{ \sum\sum -\left(\frac{m\pi}{a}\right)^2 f_m(y) f_m''(y) \sin^2 \frac{m\pi x}{a} - -\sum\sum \left(\frac{m\pi}{a}\right)^2 (f_m'(y))^2 \cos^2 \frac{m\pi x}{a} \right\} dxdy - \frac{N_{xx}}{2} \iint \sum\sum \left(\frac{m\pi}{a}\right)^2 (f_m^2(y)) \cos^2 \frac{m\pi x}{2} dxdy \tag{20}$$

Simplifying,

$$\Pi = \sum\sum \left\{ \int_0^b \frac{D}{2} \left\{ (f_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) f_m(y) + \left(\frac{m\pi}{a}\right)^4 f_m^2(y) \right\} I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 \left\{ (f_m''(y))^2 I_2 + \right.\right.$$

$$f_m(y)f_m''(y)I_1\} - \frac{N_{xx}}{2}\left(\frac{m\pi}{a}\right)^2 f_m^2(y)I_2\Big\}dy \quad (21)$$

$$I_1 = \int_0^a \sin^2 \frac{m\pi x}{a}dx \quad (22)$$

where $I_2 = \int_0^a \cos^2 \frac{m\pi x}{a}dx$ (23)

$$\frac{2\Pi}{D} = \Pi_* = \sum\sum\left\{\int_0^b \left\{(f_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y)f_m(y) + \left(\frac{m\pi}{a}\right)^4 f_m^2(y)\right\}I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 \left((f_m'(y))^2 I_2 + f_m(y)f_m''(y)I_1\right) - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 f_m^2(y)I_2\right\}dy$$

$$= \sum\sum\int_0^b F(y, f_m(y), f_m'(y), f_m''(y))dy \quad (24)$$

where $F(y, f_m(y), f_m'(y), f_m''(y))$ is the integrand in the expression for Π_*.

3.3. Euler – Lagrange differential equation for the extremum of Π_* and Π

For extremizing Π_* and hence Π we have:

$$\frac{\partial F}{\partial f_m(y)} - \frac{d}{dy}\frac{\partial F}{\partial f_m'(y)} + \frac{d^2}{dy^2}\frac{\partial F}{\partial f_m''(y)} = 0 \quad (25)$$

Differentiation of F gives:

$$\frac{\partial F}{\partial f_m(y)} = \left(\frac{m\pi}{a}\right)^4 2f_m(y)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_2 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 \quad (26)$$

$$\frac{\partial F}{\partial f_m'(y)} = 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 2f_m''(y)I_2 \quad (27)$$

$$\frac{\partial F}{\partial f_m''(y)} = \left(2f_m''(y) - 2\left(\frac{m\pi}{a}\right)^2 f_m(y)\right)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m(y)I_1 \quad (28)$$

Then, we obtain:

$$\left(\frac{m\pi}{a}\right)^4 2f_m(y)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_2 - \frac{d}{dx}\left(2(1-\mu)\left(\frac{m\pi}{a}\right)^2 2f_m''(y)I_2\right) + \frac{d^2}{dx^2}\left(2f_m''(y)I_1 - 2\left(\frac{m\pi}{a}\right)^2 f_m(y)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m(y)I_1\right) = 0$$

…(29)

Simplifying,

$$\left(\frac{m\pi}{a}\right)^4 2f_m(y)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_2 - 4(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_2 + 2f_m^{iv}(y)I_1 - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 + 2(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 = 0 \quad (30)$$

Simplifying further,

$$\left(\frac{m\pi}{a}\right)^4 2f_m(y)I_1 - 4\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 + 2f_m^{iv}(y)I_1 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_2 + 4(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 - 4(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_2 = 0 \quad (31)$$

Further simplification gives:

$$\left(\frac{m\pi}{a}\right)^4 2f_m(y)I_1 - 4\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_1 + 2f_m^{iv}(y)I_1 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_2 + 4(1-\mu)\left(\frac{m\pi}{a}\right)^2 f_m''(y)(I_1 - I_2) = 0$$

…(32)

But, $I_1 = I_2 = I_*$ (33)

Then,

$$\left(\frac{m\pi}{a}\right)^4 2f_m(y)I_* - 4\left(\frac{m\pi}{a}\right)^2 f_m''(y)I_* + 2f_m^{iv}(y)I_* - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 2f_m(y)I_* = 0 \quad (34)$$

Division by $2I_*$ yields the following differential equation:

$$\left(\frac{m\pi}{a}\right)^4 f_m(y) - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) + f_m^{iv}(y) - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 f_m(y) = 0 \quad (35)$$

Simplifying, we have the system of ordinary differential equations:

$$f_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) + \left(\left(\frac{m\pi}{a}\right)^4 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2\right)f_m(y) = 0 \quad (36)$$

4. Results

4.1. Solution of the ODE

The Euler – Lagrange differential equation is obtained as a system of homogeneous ODEs in $f_m(y)$. The ODE is solved using the methods for mathematical solutions of ODEs. By the method of trial functions, let us assume a solution for $f_m(y)$ in the exponential form:

$$f_m(y) = \exp \lambda y \quad (37)$$

wherein λ is a yet to be found parameter.

Then the ODE becomes:

$$\left[\lambda^4 - 2\left(\frac{m\pi}{a}\right)^2 \lambda^2 + \left(\left(\frac{m\pi}{a}\right)^4 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2\right)\right] e^{\lambda y} = 0 \quad \ldots(38)$$

For useful solutions, $e^{\lambda y} \neq 0$. The auxiliary equation is obtained as the fourth degree polynomial in λ, given by:

$$\lambda^4 - 2\left(\frac{m\pi}{a}\right)^2 \lambda^2 + \left(\frac{m\pi}{a}\right)^4 - \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 = 0 \quad (39)$$

The solution is obtained from simplification as:

$$\left(\lambda^2 - \left(\frac{m\pi}{a}\right)^2\right)^2 = \frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2 \quad (40)$$

$$\lambda^2 - \left(\frac{m\pi}{a}\right)^2 = \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} \quad (41)$$

$$\lambda^2 = \left(\frac{m\pi}{a}\right)^2 \pm \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} \quad (42)$$

$$\lambda^2 = \left(\frac{m\pi}{a}\right)^2 + \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} = \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} + \left(\frac{m\pi}{a}\right)^2 \quad \ldots(43)$$

$$\lambda^2 = \left(\frac{m\pi}{a}\right)^2 - \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} = -\left(-\left(\frac{m\pi}{a}\right)^2 + \sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2}\right) \quad \ldots(44)$$

$$\lambda^2 = -\left(\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} - \left(\frac{m\pi}{a}\right)^2\right) \quad (45)$$

Hence the four roots of the auxiliary polynomial are:

$$\lambda_{1m} = \left(\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} + \left(\frac{m\pi}{a}\right)^2\right)^{1/2} \quad (46)$$

and the complex conjugate roots are $\pm i\lambda_2$, where

$$\lambda_{2m} = \left(\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2} - \left(\frac{m\pi}{a}\right)^2\right)^{1/2} \quad (47)$$

i is the imaginary number, $i = \sqrt{-1}$

4.2. General solution for $f_m(y)$

The general expression for $f_m(y)$ is obtained as:

$$f_m(y) = c_{1m}\cosh\lambda_{1m}y + c_{2m}\sinh\lambda_{1m}y + c_{3m}\cos\lambda_{2m}y + c_{4m}\sin\lambda_{2m}y \quad (48)$$

where c_{1m}, c_{2m}, c_{3m} and c_{4m} are the four sets of integration constants which are obtained from the boundary conditions along the unloaded boundaries.

Case 1: Solution for CSCS (or SCSC) plates

For CSCS or SCSC plates with two simply supported edges and two clamped sides, we have from Equation (8) the boundary

$$f_m(0) = 0$$
$$f_m(b) = 0$$
$$f'_m(0) = 0$$
$$f'_m(b) = 0$$

(49)

wherein,

$$f'_m(y) = c_{1m}\lambda_{1m}\sinh\lambda_{1m}y + c_{2m}\lambda_{1m}\cosh\lambda_{1m}y - c_{3m}\lambda_{2m}\sin\lambda_{2m}y + c_{4m}\lambda_{2m}\cos\lambda_{2m}y \quad (50)$$

Enforcement of the boundary conditions yield:

$$f_m(y=0) = c_{1m} + c_{3m} = 0 \quad (51)$$

$$f'_m(y=0) = c_{2m}\lambda_{1m} + c_{4m}\lambda_{2m} = 0 \quad (52)$$

$$f_m(y=b) = c_{1m}\cosh\lambda_{1m}b + c_{2m}\sinh\lambda_{1m}b + c_{3m}\cos\lambda_{2m}b + c_{4m}\sin\lambda_{2m}b = 0 \quad (53)$$

$$f'_m(y=b) = c_{1m}\lambda_{1m}\sinh\lambda_{1m}b + c_{2m}\lambda_{1m}\cosh\lambda_{1m}b - c_{3m}\lambda_{2m}\sin\lambda_{2m}b + c_{4m}\lambda_{2m}\cos\lambda_{2m}b = 0 \quad (54)$$

In matrix form, the system of four homogeneous equations becomes:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_{1m} & 0 & \lambda_{2m} \\ \cosh\lambda_{1m}b & \sinh\lambda_{1m}b & \cos\lambda_{2m}b & \sin\lambda_{2m}b \\ \lambda_{1m}\sinh\lambda_{1m}b & \lambda_{1m}\cosh\lambda_{1m}b & -\lambda_{2m}\sin\lambda_{2m}b & \lambda_{2m}\cos\lambda_{2m}b \end{pmatrix}\begin{pmatrix} c_{1m} \\ c_{2m} \\ c_{3m} \\ c_{4m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots(55)$$

For nontrivial solutions,

$$\begin{pmatrix} c_{1m} \\ c_{2m} \\ c_{3m} \\ c_{4m} \end{pmatrix} \neq 0$$

The condition for nontrivial solution is the vanishing of the determinant of the coefficients matrix yielding the elastic stability equation as:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_{1m} & 0 & \lambda_{2m} \\ \cosh\lambda_{1m}b & \sinh\lambda_{1m}b & \cos\lambda_{2m}b & \sin\lambda_{2m}b \\ \lambda_{1m}\sinh\lambda_{1m}b & \lambda_{1m}\cosh\lambda_{1m}b & -\lambda_{2m}\sin\lambda_{2m}b & \lambda_{2m}\cos\lambda_{2m}b \end{vmatrix} = 0 \quad \ldots(56)$$

The elastic stability equation can be obtained using a simpler procedure that reduces the elastic stability problem to two equations in two unknowns. This is accomplished by expressing c_{3m} in terms of c_{1m} and c_{4m} in terms of c_{2m}.

Thus, using

$$c_{3m} = -c_{1m} \quad (57)$$

and
$$c_{4m} = -\frac{c_{2m}\lambda_{1m}}{\lambda_{2m}} \quad (58)$$

The system of homogeneous equations becomes:

$$(\cosh\lambda_{1m}b - \cos\lambda_{2m}b)c_{1m} + c_{2m}\left(\sinh\lambda_{1m}b - \frac{\lambda_{1m}}{\lambda_{2m}}\sin\lambda_{2m}b\right) = 0 \quad (59)$$

and

$$c_{1m}(\lambda_{1m}\sinh\lambda_{1m}b + \lambda_{2m}\sin\lambda_{2m}b) + c_{2m}(\lambda_{1m}\cosh\lambda_{1m}b - \lambda_{1m}\cos\lambda_{2m}b) = 0 \quad (60)$$

Hence in matrix form:

$$\begin{pmatrix} (\cosh\lambda_{1m}b - \cos\lambda_{2m}b) & \left(\sinh\lambda_{1m}b - \frac{\lambda_{1m}}{\lambda_{2m}}\sin\lambda_{2m}b\right) \\ (\lambda_{1m}\sinh\lambda_{1m}b + \lambda_{2m}\sin\lambda_{2m}b) & (\lambda_{1m}\cosh\lambda_{1m}b - \lambda_{1m}\cos\lambda_{2m}b) \end{pmatrix}\begin{pmatrix} c_{1m} \\ c_{2m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots(61)$$

For nontrivial solution, $\begin{pmatrix} c_{1m} \\ c_{2m} \end{pmatrix} \neq 0$, and the elastic stability equation is obtained as:

$$\begin{vmatrix} (\cosh\lambda_{1m}b - \cos\lambda_{2m}b) & \left(\sinh\lambda_{1m}b - \frac{\lambda_{1m}}{\lambda_{2m}}\sin\lambda_{2m}b\right) \\ (\lambda_{1m}\sinh\lambda_{1m}b + \lambda_{2m}\sin\lambda_{2m}b) & (\lambda_{1m}\cosh\lambda_{1m}b - \lambda_{1m}\cos\lambda_{2m}b) \end{vmatrix} = 0 \quad \ldots(62)$$

Expansion of Equation (62) gives:

$$(\cosh\lambda_{1m}b - \cos\lambda_{2m}b)\lambda_{1m}(\cosh\lambda_{1m}b - \cos\lambda_{2m}b) - \left(\sinh\lambda_{1m}b - \frac{\lambda_{1m}}{\lambda_{2m}}\sin\lambda_{2m}b\right)(\lambda_{1m}\sinh\lambda_{1m}b + \lambda_{2m}\sin\lambda_{2m}b) = 0 \quad \ldots(63)$$

$$\lambda_{1m}(\cosh\lambda_{1m}b - \cos\lambda_{2m}b)^2 - \left(\frac{\lambda_{2m}\sinh\lambda_{1m}b - \lambda_{1m}\sin\lambda_{2m}b}{\lambda_{2m}}\right)(\lambda_{1m}\sinh\lambda_{1m}b + \lambda_{2m}\sin\lambda_{2m}b) = 0 \quad \ldots(64)$$

$$\lambda_{1m}(\cosh^2\lambda_{1m}b + \cos^2\lambda_{2m}b - 2\cosh\lambda_{1m}b\cos\lambda_{2m}b) - \left(\frac{\lambda_{1m}\lambda_{2m}\sinh^2\lambda_{1m}b + \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b - \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b - \lambda_{1m}\lambda_{2m}\sin^2\lambda_{2m}b}{\lambda_{2m}}\right) = 0 \quad \ldots(65)$$

Multiplication by λ_{2m} gives:

$$\lambda_{1m}\lambda_{2m}(\cosh^2\lambda_{1m}b + \cos^2\lambda_{2m}b - 2\cosh\lambda_{1m}b\cos\lambda_{2m}b) - (\lambda_{1m}\lambda_{2m}\sinh^2\lambda_{1m}b + \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b - \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b - \lambda_{1m}\lambda_{2m}\sin^2\lambda_{2m}b) = 0 \quad (66)$$

Simplifying,

$$\lambda_{1m}\lambda_{2m}\cosh^2\lambda_{1m}b + \lambda_{1m}\lambda_{2m}\cos^2\lambda_{2m}b - 2\lambda_{1m}\lambda_{2m}\cosh\lambda_{1m}b\cos\lambda_{2m}b - \lambda_{1m}\lambda_{2m}\sinh^2\lambda_{1m}b - \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b + \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b + \lambda_{1m}\lambda_{2m}\sin^2\lambda_{2m}b = 0 \quad (67)$$

Simplifying again,

$$\lambda_{1m}\lambda_{2m}(\cosh^2\lambda_{1m}b - \sinh^2\lambda_{1m}b) + \lambda_{1m}\lambda_{2m}(\cos^2\lambda_{2m}b + \sin^2\lambda_{2m}b) - 2\lambda_{1m}\lambda_{2m}\cosh\lambda_{1m}b\cos\lambda_{1m}b + \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b - \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b = 0 \quad (68)$$

Using the trigonometric identities,

$$\cosh^2\lambda_{1m}b - \sinh^2\lambda_{1m}b = 1 \quad (69)$$

$$\cos^2\lambda_{2m}b + \sin^2\lambda_{2m}b = 1 \quad (70)$$

Then,

$$\lambda_{1m}\lambda_{2m}(1 + 1 - 2\cosh\lambda_{1m}b\cos\lambda_{1m}b) = \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b - \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b \quad (71)$$

Simplifying,

$$\lambda_{1m}\lambda_{2m}(2 - 2\cosh\lambda_{1m}b\cos\lambda_{1m}b) = \lambda_{2m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b - \lambda_{1m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b \quad (72)$$

Dividing by $\lambda_{1m}\lambda_{2m}$ gives:

$$2(1 - \cosh\lambda_{1m}b\cos\lambda_{1m}b) = \frac{\lambda_{2m}^2}{\lambda_{1m}\lambda_{2m}}\sinh\lambda_{1m}b\sin\lambda_{2m}b - \frac{\lambda_{1m}^2}{\lambda_{1m}\lambda_{2m}}\sin\lambda_{2m}b\sinh\lambda_{1m}b \quad \ldots(73)$$

Hence,

$$2(1 - \cosh\lambda_{1m}b\cos\lambda_{1m}b) = \frac{\lambda_{2m}}{\lambda_{1m}}\sinh\lambda_{1m}b\sin\lambda_{2m}b - \frac{\lambda_{1m}}{\lambda_{2m}}\sin\lambda_{2m}b\sinh\lambda_{1m}b \quad \ldots(74)$$

Simplifying further yields the transcendental equation:

$$2(1 - \cosh\lambda_{1m}b\cos\lambda_{1m}b) = \left(\frac{\lambda_{2m}}{\lambda_{1m}} - \frac{\lambda_{1m}}{\lambda_{2m}}\right)\sinh\lambda_{1m}b\sin\lambda_{2m}b \quad \ldots(75)$$

where λ_{1m} and λ_{2m} are expressed by Equations (46) and (47).

The characteristic elastic stability equation is a transcendental equation which is solved using computer software based iteration methods; and or manual iteration methods such as the Newton iteration, Newton – Raphson's iteration.

The solution for the elastic buckling load coefficients obtained for $\mu = 0.25$, and for aspect ratios (a/b) for CSCS (or SCSC) plates are shown in Table 1, which also shows the previous results obtained by Timoshenko [1], Onah et al [24] and Onyia et al [26]. The corresponding solutions for a Poisson's ratio $\mu = 0.30$ for various plate aspect ratios are presented in Table 2.

Table 1
Elastic buckling load coefficients of SCSC (or CSCS) Kirchhoff plate under uniform in-plane compressive load applied at the edges $x = 0$, $x = a$ for ($\mu = 0.25$)

a/b	$K(a/b) = N_{cr}b^2/\pi^2 D$ (Present study)	$K(a/b)$ Onah et al [24], Onyia et al [26], Timoshenko [1]

0.4	9.448	9.448
0.6	7.055	7.055
0.8	7.304	7.304
1.0	7.691	7.691
1.2	7.055	7.055
1.4	7.001	7.001
1.6	7.304	7.304
1.8	7.055	7.055
2	6.972	6.972

Table 2
Elastic buckling load coefficients of SCSC (or CSCS) Kirchhoff plate under uniform uniaxial in-plane compressive loads applied at $x = 0$, $x = a$ for $\mu = 0.30$

a/b	$K(a/b)$	$K(a/b)$ Onah et al [24]
0.2	27.86	27.86
0.4	9.46	9.46
0.8	7.44	7.44
1.0	7.69	7.69
$\sqrt{2}$	7.04	7.04
2	6.99	6.99
$\sqrt{6}$	7.02	7.02
3.2	6.98	6.98
$\sqrt{12}$	6.99	6.99
4.5	6.98	6.98

Case 2: Solutions for buckling of SSSS Kirchhoff plates

The boundary conditions are given alternatively using $f_m(y)$ as follows:

$$f_m(y=0)=0$$

$$f_m(y=b)=0$$

$$f_m''(y=0)=\frac{d^2 f_m(y=0)}{dy^2}=0 \qquad (76)$$

$$f_m''(y=b)=\frac{d^2 f_m(y=b)}{dy^2}=0$$

By differentiation

$$f_m''(y)=c_{1m}\lambda_{1m}^2\cosh\lambda_{1m}y+c_{2m}\lambda_{1m}^2\sinh\lambda_{1m}y-c_{3m}\lambda_{2m}^2\cos\lambda_{2m}y-c_{4m}\lambda_{2m}^2\sin\lambda_{2m}y \qquad (77)$$

Enforcement of boundary conditions gives:

$$f_m(y=0)=c_{1m}+c_{3m}=0 \qquad (78)$$

$$f_m(y=b)=c_{1m}\cosh\lambda_{1m}b+c_{2m}\sinh\lambda_{1m}b+c_{3m}\cos\lambda_{2m}b+c_{4m}\sin\lambda_{2m}b=0 \qquad (79)$$

$$f_m''(y=0)=c_{1m}\lambda_{1m}^2-c_{3m}\lambda_{2m}^2=0 \qquad (80)$$

$$f_m''(y=b)=c_{1m}\lambda_{1m}^2\cosh\lambda_{1m}b+c_{2m}\lambda_{1m}^2\sinh\lambda_{1m}b-c_{3m}\lambda_{2m}^2\cos\lambda_{2m}b-c_{4m}\lambda_{2m}^2\sin\lambda_{2m}b=0 \qquad (81)$$

From Equation (78),

$$c_{1m}=-c_{3m} \qquad (82)$$

Substitution into Equation (80) gives:

$$c_{1m}\lambda_{1m}^2+c_{1m}\lambda_{2m}^2=c_{1m}(\lambda_{1m}^2+\lambda_{2m}^2)=0 \qquad (83)$$

Hence, since $\lambda_{1m}^2+\lambda_{2m}^2\neq 0$,

$$c_{1m}=0 \qquad (84)$$

$$c_{3m}=0 \qquad (85)$$

Then the system of homogeneous equations reduces to

$$c_{2m}\sinh\lambda_{1m}b+c_{4m}\sin\lambda_{2m}b=0 \qquad (86)$$

$$c_{2m}\lambda_{1m}^2\sinh\lambda_{1m}b-c_{4m}\lambda_{2m}^2\sin\lambda_{2m}b=0 \qquad (87)$$

Or,

$$\begin{pmatrix}\sinh\lambda_{1m}b & \sin\lambda_{2m}b\\ \lambda_{1m}^2\sinh\lambda_{1m}b & -\lambda_{2m}^2\sin\lambda_{2m}b\end{pmatrix}\begin{pmatrix}c_{2m}\\ c_{4m}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad (88)$$

For useful solutions, $\begin{pmatrix}c_{2m}\\ c_{4m}\end{pmatrix}\neq 0$ the elastic buckling equation becomes:

$$\begin{vmatrix}\sinh\lambda_{1m}b & \sin\lambda_{2m}b\\ \lambda_{1m}^2\sinh\lambda_{1m}b & -\lambda_{2m}^2\sin\lambda_{2m}b\end{vmatrix}=0 \qquad (89)$$

Expanding the determinant gives:

$$-\lambda_{2m}^2\sin\lambda_{2m}b\sinh\lambda_{1m}b-\lambda_{1m}^2\sinh\lambda_{1m}b\sin\lambda_{2m}b=0 \qquad (90)$$

Simplifying,

$$-\sin\lambda_{2m}b(\lambda_{2m}^2\sinh\lambda_{1m}b+\lambda_{1m}^2\sinh\lambda_{1m}b)=0 \qquad (91)$$

or, $\sin\lambda_{2m}b\sinh\lambda_{1m}b(-\lambda_{2m}^2-\lambda_{1m}^2)=0 \qquad (92)$

Solving, for nontrivial solutions,

$$\sin\lambda_{2m}b=0 \qquad (93)$$

Hence,

$$\lambda_{2m}b=\sin^{-1}0=n\pi \qquad (94)$$

$n = 1, 2, 3, 4, \ldots$

$$\lambda_{2m}=\frac{n\pi}{b} \qquad (95)$$

From Equation (47) we have

$$\lambda_{2m}=\left(\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2}-\left(\frac{m\pi}{a}\right)\right)^{1/2}=\frac{n\pi}{b} \qquad (96)$$

Squaring both sides, we obtain:

$$\lambda_{2m}^2=\left(\frac{n\pi}{b}\right)^2=\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2}-\left(\frac{m\pi}{a}\right)^2 \qquad (97)$$

Hence, $$\sqrt{\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2}=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad (98)$$

Squaring both sides yields:

$$\frac{N_{xx}}{D}\left(\frac{m\pi}{a}\right)^2=\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \qquad (99)$$

Simplifying,

$$N_{xx}=D\left(\frac{a}{m\pi}\right)^2\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \qquad (100)$$

Then,

$$N_{xx} = D\left(\frac{a}{m}\right)^2 \frac{1}{\pi^2}\cdot\pi^4\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2 \quad (101)$$

Hence,

$$N_{xx} = D\pi^2 a^2 \frac{1}{m^2}\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2 \quad (102)$$

So,

$$N_{xx} = D\pi^2 a^2 \frac{1}{m^2}\left(\frac{m^4}{a^4}+2\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right) \quad (103)$$

Let $\frac{a}{b} = r$ (104)

$a = br$ (105)

Then,

$$N_{xx} = D\pi^2 b^2 r^2 \frac{1}{m^2}\left(\frac{m^4}{b^4r^4}+\frac{2m^2n^2}{b^2b^2r^2}+\frac{n^4}{b^4}\right) \quad (106)$$

Thus,

$$N_{xx} = D\pi^2 b^2 \frac{1}{m^2}\left(\frac{m^4}{b^4r^2}+\frac{2m^2n^2}{b^4}+\frac{n^4r^2}{b^4}\right) \quad (107)$$

Simplifying,

$$N_{xx} = \frac{D\pi^2}{b^2}\left(\frac{m^2}{r^2}+2n^2+\frac{n^4r^2}{m^2}\right) \quad (108)$$

Hence,

$$N_{xx} = K(r)\frac{D\pi^2}{b^2} = K\left(\frac{a}{b}\right)\frac{D\pi^2}{b^2} \quad (109)$$

$$K(r) = K\left(\frac{a}{b}\right) = \left(\frac{m^2}{r^2}+2n^2+\frac{n^4r^2}{m^2}\right) \quad (110)$$

For $m = n = 1$,

$$K\left(\frac{a}{b}\right) = \left(\frac{1}{r^2}+2+r^2\right) \quad (111)$$

$K(a/b)$ is tabulated for the case of simply supported edges for various values of a/b in Table 3

Table 3
Dimensionless critical elastic buckling load coefficients for the simply supported Kirchhoff plate under uniaxial uniform compression loading applied at $x = 0$, and $x = a$ for various values of a/b

a/b	(Present work) $K(a/b)$	$K(a/b)$ Iyengar [25]	$K(a/b)$ Nwoji et al [23]	$K(a/b)$ Onyia et al [27, 28]
0.1	102.01	102.01	102.01	102.01
0.2	27.04	27.04	27.04	27.04
0.3	13.201111	13.2011	13.2011	13.2011
0.4	8.41	8.41	8.41	8.41
0.5	6.25	6.25	6.25	6.25
0.6	5.137778	5.1378	5.1378	5.1378
0.7	4.530816	4.5308	4.5308	4.5308
0.8	4.2025	4.2025	4.2025	4.2025
0.9	4.044568	4.0446	4.0446	4.0446
1.0	4	4	4	4

Case 3: Solution for elastic buckling of SSSF Kirchhoff plate

The elastic buckling problem of SSSF Kirchhoff plate in Figure 4 is considered.

The boundary conditions along the edges $y = 0$ and $y = b$ are given by Equation (10).

The boundary conditions are expressed in terms of $f_m(y)$ as follows:

$$\begin{aligned} &f_m(0) = 0\\ &f_m''(0) - \mu\left(\frac{m\pi}{a}\right)^2 f_m(0) = 0\\ &f_m''(b) - \mu\left(\frac{m\pi}{a}\right)^2 f_m(b) = 0\\ &f_m'''(b) - (2-\mu)\left(\frac{m\pi}{a}\right)^2 f_m'(b) = 0 \end{aligned} \quad (112)$$

$$f_m'''(y) = c_{1m}\lambda_{1m}^3 \sinh\lambda_{1m}y + c_{2m}\lambda_{1m}^3\cosh\lambda_{1m}y + c_{3m}\lambda_{2m}^3\sin\lambda_{2m}y - c_{4m}\lambda_{2m}^3\cos\lambda_{2m}y \quad (113)$$

Enforcement of the boundary conditions give:

$$f_m(y=0) = c_{1m} + c_{3m} = 0 \quad (114)$$

$$c_{1m}\lambda_{1m}^2 - c_{3m}\lambda_{2m}^2 - \mu\left(\frac{m\pi}{a}\right)^2(c_{1m}+c_{3m}) = 0 \quad (115)$$

$$c_{1m}\lambda_{1m}^2\cosh\lambda_{1m}b + c_{2m}\lambda_{1m}^2\sinh\lambda_{1m}b - c_{3m}\lambda_{2m}^2\cos\lambda_{2m}b - c_{4m}\lambda_{2m}^2\sin\lambda_{2m}b - \mu\left(\frac{m\pi}{a}\right)^2(c_{1m}\cosh\lambda_{1m}b + c_{2m}\sinh\lambda_{1m}b + c_{3m}\cos\lambda_{2m}b + c_{4m}\sin\lambda_{2m}b) = 0 \quad (116)$$

$$c_{1m}\lambda_{1m}^3\sinh\lambda_{1m}b + c_{2m}\lambda_{1m}^3\cosh\lambda_{1m}b + c_{3m}\lambda_{2m}^3\sin\lambda_{2m}b - c_{4m}\lambda_{2m}^3\cos\lambda_{2m}b - (2-\mu)\left(\frac{m\pi}{a}\right)^2(c_{1m}\lambda_{1m}\sinh\lambda_{1m}b + c_{2m}\lambda_{1m}\cosh\lambda_{1m}b - c_{3m}\lambda_{2m}\sin\lambda_{2m}b + c_{4m}\lambda_{2m}\cos\lambda_{2m}b) = 0 \quad \ldots(117)$$

Hence,

$$c_{1m} = -c_{3m} \quad (118)$$

Then,

$$c_{1m}\left(\lambda_{1m}^2 - \mu\left(\frac{m\pi}{a}\right)^2\right) + c_{1m}\left(\lambda_{2m}^2 + \mu\left(\frac{m\pi}{a}\right)^2\right) = 0 \quad (119)$$

$$c_{1m}(\lambda_{1m}^2 + \lambda_{2m}^2) = 0 \quad (120)$$

Solving, $c_{1m} = 0$ (121)

$c_{3m} = 0$ (122)

Then the equations simplify to:

$$c_{2m}\left(\lambda_{1m}^2\sinh\lambda_{1m}b - \mu\left(\frac{m\pi}{a}\right)^2\sinh\lambda_{1m}b\right) - c_{4m}\left(\lambda_{2m}^2\sin\lambda_{2m}b + \mu\left(\frac{m\pi}{a}\right)^2\sin\lambda_{2m}b\right) = 0 \quad (123)$$

$$c_{2m}\left(\lambda_{1m}^3\cosh\lambda_{1m}b-(2-\mu)\left(\frac{m\pi}{a}\right)^2\lambda_{1m}\cosh\lambda_{1m}b\right)-$$

$$c_{4m}\left(\lambda_{2m}^3\cos\lambda_{2m}b+(2-\mu)\left(\frac{m\pi}{a}\right)^2\lambda_{2m}\cos\lambda_{2m}b\right)=0 \quad \ldots(124)$$

Hence,

$$c_{2m}\left(\lambda_{1m}^2-\mu\left(\frac{m\pi}{a}\right)^2\right)\sinh\lambda_{1m}b-c_{4m}\left(\lambda_{2m}^2+\mu\left(\frac{m\pi}{a}\right)^2\right)\sin\lambda_{2m}b=0 \quad \ldots(125)$$

$$c_{2m}\lambda_{1m}\left(\lambda_{1m}^2-(2-\mu)\left(\frac{m\pi}{a}\right)^2\right)\cosh\lambda_{1m}b-$$

$$c_{4m}\lambda_{2m}\left(\lambda_{2m}^2+(2-\mu)\left(\frac{m\pi}{a}\right)^2\right)\cos\lambda_{2m}b=0 \quad (126)$$

It can be shown that

$$\lambda_{1m}^2-\mu\left(\frac{m\pi}{a}\right)^2=\lambda_{2m}^2+(2-\mu)\left(\frac{m\pi}{a}\right)^2=\beta_{1m} \quad (127)$$

$$\lambda_{2m}^2+\mu\left(\frac{m\pi}{a}\right)^2=\beta_{2m}=\lambda_{1m}^2-(2-\mu)\left(\frac{m\pi}{a}\right)^2 \quad (128)$$

Then,

$$c_{2m}\beta_1\sinh\lambda_{1m}b-c_{4m}\beta_2\sin\lambda_{2m}b=0 \quad (129)$$

$$c_{2m}\lambda_{1m}\beta_{2m}\cosh\lambda_{1m}b+c_{4m}\lambda_{2m}(-\beta_{1m})\cos\lambda_{2m}b=0 \quad (130)$$

In matrix form,

$$\begin{pmatrix}\beta_{1m}\sinh\lambda_{1m}b & -\beta_{2m}\sin\lambda_{2m}b\\ \lambda_{1m}\beta_{2m}\cosh\lambda_{1m}b & -\beta_{1m}\lambda_{2m}\cos\lambda_{2m}b\end{pmatrix}\begin{pmatrix}c_{2m}\\ c_{4m}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \ldots(131)$$

For nontrivial solutions, $\begin{pmatrix}c_{2m}\\ c_{4m}\end{pmatrix}\neq 0.$

The characteristic elastic buckling equation is obtained as:

$$\begin{vmatrix}\beta_{1m}\sinh\lambda_{1m}b & -\beta_{2m}\sin\lambda_{2m}b\\ \lambda_{1m}\beta_{2m}\cosh\lambda_{1m}b & -\beta_{1m}\lambda_{2m}\cos\lambda_{2m}b\end{vmatrix}=0 \quad (132)$$

Expanding yields:

$$-\beta_{1m}^2\lambda_{2m}\sinh\lambda_{1m}b\cos\lambda_{2m}b+\lambda_{1m}\beta_{2m}^2\cosh\lambda_{1m}b\sin\lambda_{2m}b=0 \quad \ldots(133)$$

Dividing by $\cosh\lambda_{1m}b\cos\lambda_{2m}b,$ gives:

$$\lambda_{1m}\beta_{2m}^2\tan\lambda_{2m}b-\lambda_{2m}\beta_{1m}^2\tanh\lambda_{1m}b=0 \quad (134)$$

or, $\lambda_{2m}\beta_{1m}^2\tanh\lambda_{1m}b=\lambda_{1m}\beta_{2m}^2\tan\lambda_{2m}b$ (135)

The solution to the transcendental equation are obtained using computer software based iteration methods based on Newton's methods or Newton – Raphson's methods. The buckling loads corresponding to the first two buckling modes and for a Poisson's ratio $\mu = 0.25$ are found for various aspect ratios and presented in Table 4. Table 4 also presents benchmark solutions found previously by Timoshenko [1].

Table 4

Elastic buckling load coefficients for SSSF Kirchhoff plate subject to compressive load N_{xx} along the boundaries $x = 0$, and $x = a$, for varying aspect ratios (a/b) and for Poisson's ratio $\mu = 0.25$.

a/b	Present study		Oguaghamba & Ike [29]		Timoshenko [1] $K(a/b)$ for $m = 1$	% Difference between present results for $n = 1$ and Timoshenko [1]
	$K(a/b)$ for $m = 1$	$K(a/b)$ for $m = 2$	$K(a/b)$ for $m = 1$	$K(a/b)$ for $m = 2$		
0.4	6.6367	25.2899	6.6367	25.2899		
0.6	3.1921	11.4675	3.1921	11.4675		
0.8	1.9894	6.3667	1.9894	6.3667		
1.0	1.4342	4.4036	1.4342	4.4036	1.44	–0.403
1.5	0.8880	2.2022	0.8880	2.2022		
2	0.6979	1.4342	0.6979	1.4342	0.698	–0.0205
2.5	0.6104	1.0798	0.6104	1.0798	0.610	0.0656
3	0.5630	0.8879	0.5630	0.8879	0.564	–0.1773
3.5	0.5343	0.7726	0.5343	0.7726		
4	0.5161	0.6979	0.5161	0.6979	0.517	–0.1741
4.5	0.5034	0.6469	0.5034	0.6469		
5	0.4944	0.6104	0.4944	0.6104		
5.5	0.4877	0.5835	0.4877	0.5835		
6	0.4826	0.5630	0.4826	0.5630		

5. Discussion

The Kantorovich variational method has been successfully used in this work for the stability problems of rectangular Kirchhoff plates with two simply supported boundaries and subjected to uniaxial uniform compressive load. Three different cases of edge supports the unloaded sides considered are:

(i) the unloaded sides are both clamped (SCSC or CSCS plate),
(ii) the unloaded edges are both simply supported hence the plate, is SSSS plate, and
(iii) the side $y = 0$ is on simple support while the side $y = b$ is free, and the plate is SSSF plate.

The general problem of elastic buckling of thin plates is presented in variational form as the problem of minimization of the functional (Equation (6)) for the elastic thin plate problem. For the specific considered problem, the total potential energy to be minimized is presented from a simplification of Equation (6) as Equation (7). By the Kantorovich variational method adopted in this work, the buckling deflection is assumed in variable – separable form as an infinite series as given by Equation (11) where $g_m(x)$, the coordinate (basis) functions in the x coordinate direction is chosen to apriori satisfy all the Dirichlet conditions at the simply supported boundaries. Thus the buckled deflection was considered as the infinite series in Equation (12) where $f_m(y)$ is an unknown function of the y coordinate variable. The elastic thin plate buckling problem hence simplifies to finding the unknown function $f_m(y)$ that extremizes (minimizes) the functional Π and simultaneously satisfies all conditions along the boundaries $y = 0$, and $y = b$ for each of the three cases considered in this study.

Since the buckled deflection function $w(x, y)$ is expressed in terms of $f_m(y)$ as given by Equation (12), the functional is expressed using $f_m(y)$ and its derivatives by substitution of $w(x, y)$ and the corresponding partial derivatives to obtain after simplifications, Equation (21). The modified total potential energy functional Π_* is given as Equation (24); where the integrand contains $f_m(y)$, $f_m'(y)$ and $f_m''(y)$. It is easy to observe that the extremum or minimum of Π_* and Π would give identical results.

The Euler – Lagrange differential equation given by Equation (25) is used to obtain the conditions for functional to be minimized. The Euler – Lagrange differential equation for extremizing the total potential energy functional is found as Equation (29). Further simplifications of Equation (29) gives the system of ODEs – Equation (36) with $f_m(y)$ as the unknown function desired to be found.

A general solution to the Euler – Lagrange ODE is obtained via trial functions by assuming a solution for $f_m(y)$ of the exponential form as in Equation (37). This reduced the ODE to the algebraic problem in Equation (38). The conditions for nontrivial solutions gave the auxiliary or homogeneous equation expressed by the polynomial – Equation (39). The solution of the auxiliary polynomial gave the four roots in Equations (46) and (47). The general solution for $f_m(y)$ was thus found as Equation (48). The elastic buckling loads are determined for each of the considered cases of edge supports by the enforcement of the appropriate boundary conditions.

5.1. Discussion on elastic buckling of SCSC (or CSCS) Kirchhoff plates

For CSCS (or SCSC) plates where the edges $y = 0$ and $y = b$ are both clamped, Equation (8) is used to obtain the system of four equations presented as Equation (49).

Enforcement of boundary conditions along the unloaded edges gave the system of four homogeneous equations – Equation (51), (52), (53) and (54), which were expressed as Equation (55).

The nontrivial solution rule of the system of equations is used to express the elastic stability (buckling) problems as Equation (56). Expansion of Equation (56) would be difficult, and hence a simplified procedure that reduces the elastic stability problem to two unknowns is used. This is accomplished by elimination of two constants of integration for the system of equations using Equations (57) and (58); yielding the system of homogeneous equations as Equations (59) and (60) or Equation (61) in matrix form. The condition for nontrivial solutions then gave the elastic stability equation as Equation (62).

Upon expansion of the determinant in Equation (62) and simplifications, the elastic buckling equation for CSCS (or SCSC) Kirchhoff plates was obtained as Equation (75); which is a transcendental equation. Equation (75) is solved using computer software and iteration methods for $\mu = 0.25$ and $\mu = 0.30$ and for plate aspect ratios between 0.4 and 2, and presented in Tables 1 and 2. Table 1 shows that the presented results agree with previous results presented by Timoshenko [1], Onah et al [24] and Onyia et al [26].

Table 1 shows that the dimensionless elastic buckling coefficients for CSCS Kirchhoff plates with $\mu = 0.25$ under in-plane uniform compressive loading obtained using the Kantorovich variational method are identical with past results obtained by Timoshenko [1], Onah et al [24], and Onyia et al [26].

Table 2 reveals that $K(a/b)$ for CSCS thin plates (with $\mu = 0.30$) under in-plane uniform compressive loading obtained using the present method are exactly the same as the solution obtained by Onah et al [24].

5.2. Discussion on elastic buckling of SSSS Kirchhoff plates

The boundary conditions for the case of simply supported edges $y = 0$ and $y = b$ were expressed in terms of $f_m(y)$ as Equation (76). Enforcement of the boundary conditions yielded a system of homogeneous Equations – Equations (78) – (81). The solution of Equations (82) and (83) gave the two unknown constants of integration as Equations (84) and (85). The system of homogeneous equations simplified to Equations (86) and (87) presented in matrix form as Equation (88).

The condition for nontrivial solutions gave the characteristic elastic buckling equation for SSSS Kirchhoff plates as Equation (89), which upon expansion and simplification gave Equation (92). The solution of the elastic buckling equation for nontrivial solutions was found for SSSS thin plates as Equation (95) which yielded the elastic buckling load expression given by Equation (108) for any buckling modes m, n and any plate aspect ratio, r. The elastic buckling load coefficient $K(a/b)$ was found as Equation (110) and for the buckling modes $m = 1$, $n = 1$, $K(a/b)$ was found as Equation (111). $K(a/b)$ is tabulated for SSSS Kirchhoff plates for various aspect ratios (a/b) and for $m = n = 1$ and presented in Table 3 which also shows that the presented results agree with previously obtained solutions for $K(a/b)$ by Iyengar [25] and Nwoji et al [23].

Table 3 demonstrates that the critical elastic buckling coefficients $K(a/b)$ for SSSS Kirchhoff plate are exact and for a/b lying between 0.1 and 1.0, $K(a/b)$ is identical with the previous results of Nwoji et al [23], Iyengar [25], and Onyia et al [27, 28].

5.3. Discussion on elastic buckling of SSSF Kirchhoff plate

The boundary conditions for the SSSF Kirchhoff plate along the $y = 0$ and $y = b$ edges are given in Equation (10); and expressed in terms of $f_m(y)$ as Equation (112). Enforcing the boundary conditions yielded a set of four equations – Equations (114) – (117). The system of four equations simplify because two constants of integration are found to vanish from Equations (121) and (122) obtained from solving simultaneously Equations (114) and (115). The system of homogeneous equations then simplified to Equations (123) and (124) which was simplified to the matrix form in Equation (131). The characteristic elastic buckling equation obtained for nontrivial solutions is given by Equation (132); which upon expansion and simplification gave Equation (135).

Equation (135) which is the characteristic elastic buckling equation for SSSF Kirchhoff plates is a transcendental equation which was solved by iteration methods for $\mu = 0.25$ and $\mu = 0.30$ and for various aspect ratios, r. The elastic buckling load coefficients for SSSF Kirchhoff plate subject to uniform uniaxial compressive load N_{xx} along the simply supported edges for $m = 1$ and $m = 2$ and for various aspect ratios and for Poisson's ratio $\mu = 0.25$ are shown in Table 4 which also shows close agreement with the previous results obtained by Timoshenko [1]. Identical results were previously obtained by Oguaghamba and Ike [29] using Galerkin-Vlasov variational method for the elastic stability of SSSF plates.

For the case of elastic buckling of SSSF Kirchhoff plate subjected to uniform compressive load, along the simply supported edges, the Kantorovich variational results presented in Table 4 (for $\mu = 0.25$) show that for the first buckling mode, the present results agree remarkably well with previous results obtained by Timoshenko [1] with relative difference of less than 0.41% in absolute terms. Table 4 further shows that $K(a/b)$ for $m = 1$ and $m = 2$ are identical with previous results obtained using the Galerkin-Vlasov method by Oguaghamba and Ike [29].

6. Conclusion

In conclusion:

(i) The Kantorovich variational method has been successfully used in this paper to solve the elastic buckling problem of Kirchhoff plate under uniform compressive loading applied at the two opposite simply supported edges $x = 0$ and $x = a$ for the cases where the edges $y = 0$ and $y = a$ are clamped; the edges $y = 0$ and $y = b$ are simply supported and the edge $y = 0$ is simply supported and the edge $y = b$ is free.

(ii) The Kantorovich variational method adopted is based on seeking to obtain a minimized energy functional Π with respect to the unknown deflection function where the deflection function is assumed as the infinite series of a product of unknown function $f_m(y)$ and known function $g_m(y)$ considered as the sinusoidal functions that satisfy the Dirichlet boundary conditions along the two opposite simply supported edges $x = 0$ and $x = a$.

(iii) The assumption of the deflection expression in variable – separable form as the infinite series of products of $f_m(y)$ and $g_m(x)$ where $g_m(x)$ is known simplifies the total potential energy functional to a functional dependent on the y coordinate variable, the unknown function $f_m(y)$ and derivatives of $f_m(y)$ ($f_m'(y)$ and $f_m''(y)$).

(iv) Euler – Lagrange differential equations which are the conditions necessary for a minimization of the total potential energy functional are applied to obtain the differential equation for $f_m(y)$ to yield a minimum value of Π. The Euler – Lagrange differential equation obtained for the presented elastic buckling problem is a system of ODEs of the fourth order.

(v) Methods for solving ODEs and system of ODEs – differential (D) operator methods, trial function methods, etc – are used for the resulting Euler – Lagrange equation obtained.

(vi) The method of trial functions reduces the fourth order ODE to a fourth degree polynomial as the auxiliary or characteristic equation required to be satisfied for nontrivial solutions for an assumption of $f_m(y)$ in exponential form.

(vii) The four eigenvalues (roots) of the auxiliary polynomial are used to obtain the general solution for $f_m(y)$ in terms of hyperbolic functions and trigonometric functions with four unknown integration constants which agrees with the fourth order nature of the ODEs.

(viii) Enforcement of boundary conditions are used to find the characteristic stability equations for each of the considered three cases of edge support conditions along the unloaded boundaries.

(ix) The characteristic stability equations found for the considered three cases of edge supports along $y = 0$ and $y = b$ were found to be exact solutions to the elastic buckling problem, and yields exact solutions for the corresponding boundary conditions considered.

(x) The characteristic elastic buckling equations obtained for SCSC (or CSCS) and SSSF Kirchhoff plates are transcendental equations which are difficult to solve for in closed form, but are solved by iteration techniques or computer software tools.

(xi) The characteristic elastic buckling equation obtained for SSSS Kirchhoff plates is solved in closed form for any given buckling mode m, n and the solutions are identical with solutions obtained previously.

(xii) The present solutions obtained for SCSC (or CSCS) Kirchhoff plates are identical with previously obtained solutions by Onah et al [24] who used the single Fourier sine integral transform method, Onyia et al [26] who implemented the Galerkin – Kantorovich method for the elastic stability analysis of thin rectangular SCSC plates.

Conflict of Interest

The authors hereby declare that they have no conflict of interest in the publication of the paper.

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