STODOLA-VIANELLO ITERATION METHOD FOR FREE TORSIONAL VIBRATION ANALYSIS OF MONOSYMMETRIC BOX-BEAM BRIDGES

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Abstract
Box-beam bridges are used in large spans and wider decks due to their high strength and greater torsional and flexural stiffness. They are usually prone to vibration due to moving vehicular traffic. Their eigenfrequency analysis is a crucial aspect of their design in order to ensure that the natural frequency is not close to the excitation frequency to avert resonance failures. The free torsional vibration equation is a fourth order partial differential equation (PDE) with variable parameters. For prismatic cross-sections, homogeneous and isotropic materials the governing PDE have constant parameters. This paper explores the Stodola-Vianello iteration method (SVIM) for solving the PDE for isotropic, homogeneous prismatic box-beams. Harmonic response is assumed, decoupling the PDE to two equations, one in terms of time and the second an ordinary differential equation (ODE) in terms of space coordinates. The Stodola-Vianello method is used by the method of four successive integrations to express the ODE as an algebraic iteration problem with four constants of integration. The four boundary conditions are used to solve for the four constants of integration, thus making the problem determinate. Application of the boundary conditions results in the full determination of the iteration equations. For simply supported boundaries studied in the work, a trigonometric buckling shape function that satisfies all the boundary conditions is employed in the SVIM formula to obtain the next bucking modal shape function. The requirement for convergence is then used to establish the characteristic buckling equation from which the eigenvalues are obtained. The solution to the characteristic buckling equation gave the exact mathematical expression for the natural torsional frequency at the *n*th vibration mode. The frequencies are determined for the first eight vibration modes and compared with previously obtained values. It was found that the natural frequencies are identical with previously found values of the exact natural frequencies. The natural frequency obtained is exact because it satisfies the PDE and the boundary conditions at all points in the domain.

Keywords: Stodola-Vianello iteration method, monosymmetric beam, free torsional vibration, torsional vibration frequency, eigenvalue.

1. Introduction

Box-girder bridges are used for cases of large spans and wider decks because of their higher strength and greater torsional and flexural stiffnesses [1]. The natural vibration characteristics of box-girder bridges including the model parameters such as mode shape and natural frequencies need to be studied to comprehensively understand its fundamental dynamic behaviour [2].

The natural frequency analysis of box-girder bridges is vital to their design in order to determine their dynamic response and ensure that the natural frequency is not close to the excitation frequency in order to avoid resonance. Every structural system has a specific pattern of vibration, called mode shape, under a specific frequency known as the natural frequency. Different modes have different natural frequencies. Modes depend upon the material properties like inertial, damping and stiffness and also on the boundary conditions of the structures.

Free vibrations occur naturally with no external excitation applied and responsible for the vibration. Free vibrations start with some energy input but dies away with time as the energy is dissipated. The natural frequencies and vibration modes depend upon the geometrical, inertial, elasticity and damping properties of the vibrating box-girder beam system.

Finite element method (FEM) was used by Shaikh and Nallasivam [3] to determine the free vibration responses of a box-girder bridge along with the railway subtrack system. Their study applied the non-closed form FEM-based ANSYS software. Agarwal et al [1] used the FEM for the free vibration analysis of simply supported reinforced concrete box girder bridges. They found the fundamental frequencies for straight, curved, skewed and skew-curved box-girder bridges. They also found that the fundamental frequencies of skew-curved bridges were more than those for straight bridges making skew bridges more preferred than straight bridges.

Tang and Zhu [4] have studied the vibration analysis of composite box-girder beam bridges. Verma and Nallasivam [5] studied the natural vibration of thin walled concrete box-girder bridge using an experimental model. Verma et al [2] used the FEM to study the free vibration analysis of thin-walled box-girder bridge. Zhang et al [6] investigated the free vibration behaviour of thin-walled rectangular box beam using generalized coordinates. Ramkumar and Kang [7] have also used the FEM to study the buckling and vibration analysis of thin-walled box beams.

Hannewald [8] applied the finite difference method (FDM) to the eigenfrequency analysis of the torsional frequencies of box-beam bridge. The researcher considered harmonic vibrations of the system which decoupled the space and time variables in the governing partial differential equation resulting in an ordinary differential equation (ODE) for the dynamic problem. The FDM was used to discretize the ODE such that the ODE becomes a system of algebraic equations written for each grid point in the finite difference grid representation of the vibrating beam.

The resulting algebraic equations are solved subject to the boundary conditions to yield the natural frequencies of vibration. Hannewald [8] also presented the analytical solutions for the refined torsional vibrating frequencies and found that the frequencies obtained using the FDM were close to the analytical solutions with differences of -1.68% for a four FDM grid point solution (for $n = 1$); - 0.524% for a FDM grid with $N = 8$, and -0.165% for FDM grid point with $N = 16$.

Literature review reveals that the Stodola-Vianello iteration method (SVIM) has not been applied to the solution of the free torsional vibration equation of box beams. This paper applies the Stodola-Vianello iteration method to the eigenfrequency determination of the box beam problem. The viability of the Stodola-Vianello iteration for solving the boundary value problems of buckling and vibration have been illustrated by the previous applications of the SVIM to Euler column buckling by Ofondu et al [9]; buckling of beam on elastic foundations by Ike et al [10], [11], and Ike [12], [13] and [14].

2. Governing Partial Differential Equation of Motion

The governing partial differential equation (GPDE) of dynamic equilibrium of a monosymmetric beam with closed cross-section shown in Figure 1 is given by Hannewald [7] as:

$$\frac{\partial^2}{\partial x^2}\left(C_w(x)\frac{\partial^2\phi(x,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(C(x)\frac{\partial\phi(x,t)}{\partial x}\right)+$$

$$\rho I_p(x)\frac{\partial^2\phi(x,t)}{\partial t^2}=M_e(x,t) \qquad (1)$$

wherein, x is the longitudinal axial coordinate.

$C_w(x)$ is the warping stiffness of the beam with units of kNm^4 where there is varying warping stiffness for non-homogeneous beam materials.

$C(x)$ is the torsional stiffness in kNm^2 where there is varying torsional stiffness for non-homogeneous beam materials.

$\rho I_p(x)$ is the mass moment of inertia per unit length.

ρ is the mass density in kg/m^3; I_p is the polar moment of inertia and $\phi(x,t)$ is the angle of twist or torsional angle; t is time.

$M_e(x,t)$ is the externally applied twisting moment in kNm/m.

$M_R(x,t)$ is the intensity of the applied distributed torque.

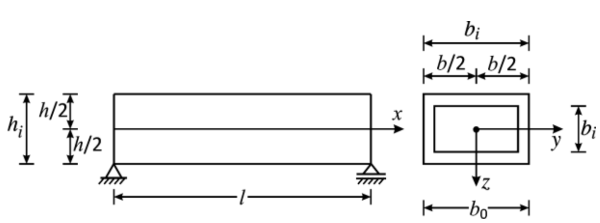

Figure 1 Monosymmetric box-beam bridge with simple supports at the ends

The warping stiffness $C_w(x)$ is expressed generally as:

$$C_w = E(x)I_w \qquad (2)$$

where $E(x)$ is the Young's modulus of elasticity in MPa.

I_w is the warping constant in m^6.

The torsional stiffness $C(x)$ is expressed generally as:

$$C(x) = G(x)J \qquad (3)$$

where $G(x)$ is the shear modulus or modulus of rigidity.

J is the torsional moment of inertia in m^4 or Saint Venant torsional constant.

I_p is given by

$$I_p = I_{yy} + I_{zz} \qquad (4)$$

where I_{yy} is moment of inertia about yy axis

I_{zz} is moment of inertia about zz axis.

Hence Equation (1) is expressed as:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I_w\frac{\partial^2\phi}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(G(x)J\frac{\partial\phi}{\partial x}\right)+\rho(x)I_p\frac{\partial^2\phi}{\partial t^2}=M_e(x,t) \qquad \ldots(5)$$

For free vibrations, the external excitation vanishes and the equation reduces to the homogeneous equation:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I_w\frac{\partial^2\phi}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(G(x)J\frac{\partial\phi}{\partial x}\right)+\rho I_p\frac{\partial^2\phi}{\partial t^2}=0 \qquad (6)$$

For prismatic cross-sections I_w, J and I_p are constants. For homogeneous beams, E, G are constants. Then the GPDE is:

$$\frac{\partial^2}{\partial x^2}\left(EI_w\frac{\partial^2\phi}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(GJ\frac{\partial\phi}{\partial x}\right)+\rho I_p\frac{\partial^2\phi}{\partial t^2}=0 \qquad (7)$$

Simplifying, the GPDE for homogeneous beams with prismatic cross-sections becomes the fourth order PDE with constant parameters.

$$EI_w\frac{\partial^4\phi}{\partial x^4}-GJ\frac{\partial^2\phi}{\partial x^2}+\rho I_p\frac{\partial^2\phi}{\partial t^2}=0 \qquad (8)$$

For harmonic torsional vibrations, it is assumed that the response is harmonic and $\phi(x,t)$ is considered as:

$$\phi(x,t)=\sum_{n=1}^{\infty}F_n(x)e^{i\omega_n t} \qquad (9)$$

where $F_n(x)$ is the nth torsional vibration modal shape function and ω_n is the angular frequency of torsional vibrations for the nth mode

$$i=\sqrt{-1} \qquad (10)$$

Then

$$\frac{\partial^2\phi(x,t)}{\partial t^2}=\sum_n^{\infty}\frac{\partial^2}{\partial t^2}F_n(x)e^{i\omega_n t}=\sum_n^{\infty}(i\omega)^2F_n(x)e^{i\omega_n t}$$

$$=\sum_n^{\infty}-\omega_n^2F_n(x)e^{i\omega_n t} \qquad (11)$$

$$\frac{\partial^2\phi(x,t)}{\partial x^2}=\sum_n^{\infty}\frac{\partial^2}{\partial x^2}F_n(x)e^{i\omega_n t}=\sum_n^{\infty}\frac{d^2}{dx^2}F_n(x)e^{i\omega_n t} \qquad (12)$$

$$\frac{\partial^4\phi(x,t)}{\partial x^4}=\sum_n^{\infty}\frac{\partial^4}{\partial x^4}F_n(x)e^{i\omega_n t}=\sum_n^{\infty}\frac{d^4}{dx^4}F_n(x)e^{i\omega_n t} \qquad (13)$$

The GPDE is then simplified to

$$\sum_n^{\infty}\left\{EI_w\frac{d^4}{dx^4}F_n(x)e^{i\omega_n t}-GJ\frac{d^2}{dx^2}F_n(x)e^{i\omega_n t}-\right.$$

$$\rho I_p \omega^2 F_n(x) e^{i\omega_n t} \Big\} = 0 \tag{14}$$

Factorizing,

$$\sum_n^{\infty} \left(EI_w \frac{d^4 F_n(x)}{dx^4} - GJ \frac{d^2}{dx^2} F_n(x) - \rho I_p \omega_n^2 F_n(x) \right) e^{i\omega_n t} = 0$$

…(15)

For nontrivial solutions, $e^{i\omega t} \neq 0$, hence

$$EI_w F_n^{iv}(x) - GJ F_n''(x) - \rho I_p \omega^2 F_n(x) = 0 \tag{16}$$

where $F_n''(x) = \dfrac{d^2 F_n(x)}{dx^2}$ (17a)

$$F_n^{iv}(x) = \frac{d^4 F_n(x)}{dx^4} \tag{17b}$$

3. Methodology

Dividing Equation (16) by EI_w gives:

$$F_n^{iv}(x) = \frac{GJ}{EI_w} F_n''(x) + \frac{\rho I_p \omega^2 F_n(x)}{EI_w} \tag{18}$$

Integrating once,

$$F_n'''(x) = \int_0^x \left(\frac{GJ}{EI_w} F_n''(x) + \frac{\rho I_p \omega^2 F_n(x)}{EI_w} \right) dx \tag{19}$$

$$F_n'''(x) = \frac{GJ}{EI_w} F_n'(x) + \frac{\rho I_p \omega^2}{EI_w} \int_0^x F_n(x)\, dx + c_1 \tag{20}$$

where c_1 is an integration constant.
Integrating again,

$$F_n''(x) = \frac{GJ}{EI_w} F_n(x) + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x F_n(x)\, dxdx + c_1 x + c_2 \tag{21}$$

where c_2 is the second integration constant.
Integrating again,

$$F_n'(x) = \frac{GJ}{EI_w} \int_0^x F_n(x)\, dx + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x \int_0^x F_n(x)\, dxdxdx + \frac{c_1 x^2}{2} + c_2 x + c_3 \tag{22}$$

where c_3 is the third integration constant.
Integrating the fourth time gives:

$$F_n(x) = \frac{GJ}{EI_w} \int_0^x \int_0^x F_n(x)\, dxdx + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x \int_0^x \int_0^x F_n(x)\, dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \tag{23}$$

where c_4 is the fourth integration constant.
Hence the Stodola-Vianello iteration formula becomes:

$$F_{n+1}''(x) = \frac{GJ}{EI_w} F_n(x) + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x F_n(x)\, dxdx + c_1 x + c_2 \tag{24}$$

$$F_{n+1}(x) = \frac{GJ}{EI_w} \int_0^x \int_0^x F_n(x)\, dxdx + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x \int_0^x \int_0^x F_n(x)\, dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4$$

...(25)

By the Stodola-Vianello iteration method, the four constants of integration c_1, c_2, c_3 and c_4 are determined using the boundary conditions.
For simply supported conditions at the beam ends $x = 0$, $x = l$, the boundary conditions on $F(x)$ are:

$$F_n(x = 0) = F_n''(x = 0) = 0 \tag{26a}$$

$$F_n(x = l) = F_n''(x = l) = 0 \tag{26b}$$

Hence a suitable exact modal shape function which satisfies the boundary conditions for the nth vibration mode is:

$$F_n(x) = \sin \frac{n\pi x}{l} \tag{27}$$

The SVIM equations are:

$$F_{n+1}''(x) = \frac{GJ}{EI_w} \sin \frac{n\pi x}{l} + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x \sin \frac{n\pi x}{l}\, dx + c_1 x + c_2$$

…(28)

$$F_{n+1}(x) = \frac{GJ}{EI_w} \int_0^x \int_0^x \sin \frac{n\pi x}{l}\, dxdx + \frac{\rho I_p \omega^2}{EI_w} \int_0^x \int_0^x \int_0^x \int_0^x \sin \frac{n\pi x}{l} dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4$$

...(29)

4. Results

Evaluating the integrals yields:

$$F_{n+1}''(x) = \frac{GJ}{EI_w} \sin \frac{n\pi x}{l} - \frac{\rho I_p \omega^2}{EI_w} \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l}\, dx + c_1 x + c_2$$

…(30)

$$F_{n+1}(x) = \frac{-GJ}{EI_w} \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l} + \frac{\rho I_p \omega^2}{EI_w} \left(\frac{l}{n\pi} \right)^4 \sin \frac{n\pi x}{l} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \tag{31}$$

Using the boundary conditions

$$F_{n+1}(x = 0) = c_4 = 0 \tag{32a}$$

$$F_{n+1}''(x = 0) = c_2 = 0 \tag{32b}$$

$$F_{n+1}''(x = l) = \frac{GJ}{EI_w} \sin n\pi - \frac{\rho I_p \omega^2}{EI_w} \left(\frac{l}{n\pi} \right)^2 \sin n\pi + c_1 l = 0$$

…(33)

$$c_1 = 0 \tag{34}$$

$$F_{n+1}(x=l) = \frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \sin n\pi + \frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4 \sin n\pi + c_3 l = 0 \quad (35)$$

$$c_3 = 0 \quad (36)$$

Hence,

$$F_{n+1}(x) = \frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + \frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi x}{l} \quad \ldots(37)$$

$$F_{n+1}(x) = \left(\frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 + \frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4\right)F_n(x) \quad (38)$$

For convergence at the *n*th buckling mode,

$$F_{n+1}(x) = F_n(x) \quad (39)$$

Hence,

$$\sin\frac{n\pi x}{l} = \left(\frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 + \frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4\right)\sin\frac{n\pi x}{l} \quad (40)$$

The characteristic eigenvalue equation is

$$1 = \left(\frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 + \frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4\right) \quad (41)$$

Rearranging gives

$$\frac{\rho I_p \omega^2}{EI_w}\left(\frac{l}{n\pi}\right)^4 = 1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \quad (42)$$

$$\frac{\rho I_p \omega^2}{EI_w} = \left(\frac{n\pi}{l}\right)^4\left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right) \quad (43)$$

$$\frac{\rho I_p \omega^2}{EI_w} = \left(\frac{n\pi}{l}\right)^4 + \frac{GJ}{EI_w}\left(\frac{n\pi}{l}\right)^2 \quad (44)$$

$$\frac{\rho I_p \omega_n^2}{EI_w} = \left(\frac{n\pi}{l}\right)^2\left(\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{EI_w}\right) \quad (45)$$

Solving for ω_n^2 gives:

$$\omega_n^2 = \frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2\left(\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{EI_w}\right) \quad (46)$$

$$\omega_n^2 = \left(\frac{n\pi}{l}\right)^2\left(\frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{\rho I_p}\right) \quad (47)$$

Taking the square root of both sides gives:

$$\omega_n = \sqrt{\left(\frac{n\pi}{l}\right)^2\left(\frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{\rho I_p}\right)} \quad (48)$$

$$\omega_n = \frac{n\pi}{l}\sqrt{\frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{\rho I_p}} \quad (49)$$

But $\omega_n = 2\pi f_n$ (50)

$$f_n = \frac{\omega_n}{2\pi} = \frac{n\pi}{2\pi l}\sqrt{\frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{\rho I_p}} \quad (51)$$

$$f_n = \frac{n}{2l}\sqrt{\frac{EI_w}{\rho I_p}\left(\frac{n\pi}{l}\right)^2 + \frac{GJ}{\rho I_p}} \quad (52)$$

Parametric Studies

The Stodola-Vianello iteration method is applied to the free torsional vibration analysis of the box beam bridge structure studied and solved by Hannewald [8] using the analytical method and the finite difference method..

For the symmetrical box-beam bridge structure with closed cross-section, the parameters are

$$\begin{aligned} &E = 36000\text{MPa}, \\ &\rho = 2500\text{kg/m}^3 \\ &I_w = 146.1999\text{m}^6, \quad \mu = 0.20 \\ &I_p = 609.9098\text{m}^4 \\ &J = 32.0042\text{m}^4 \\ &l = 78\text{m} \end{aligned} \quad (53)$$

Then,

$$G = \frac{E}{2(1+\mu)} = \frac{36000}{2(1+0.20)}\text{MPa} = 15000\text{MPa} \quad (54)$$

$$C_w = EI_w = 36000\text{MPa} \times 146.1999\text{m}^6 \quad (55a)$$

$$EI_w = 5,263,196.4\text{MNm}^4 \quad (55b)$$

$$C = GJ = 15000\text{MPa} \times 32.0042\text{m}^4 = 480,063\text{MNm}^2 \quad (56)$$

$$\rho I_p = 2500\text{kg/m}^3 \times 609.9098\text{m}^4 \quad (57)$$

$$\rho I_p = 1,524,774.5\text{Nm}^6 \quad (58)$$

$$\rho I_p = 1.5248\text{MNm}^6 \quad (59)$$

Hence,

$$f_n = \frac{n}{2\times 78}\sqrt{\frac{5,263,196.4}{1.5248}\left(\frac{n\pi}{78}\right)^2 + \left(\frac{480,063}{1.5248}\right)} \quad (60)$$

where $n = 1, 2, 3\ldots$

f_n is evaluated for $n = 1, 2, 3, 4, 5, 6, 7$ and 8 and the values obtained are tabulated in Table 1 and compared with previously obtained values from Hannewald [8] who presented the analytical solution and the finite difference solution.

Table 1: Torsional frequencies of various vibration modes for free torsional vibration of a box-beam bridge

Torsional vibration mode	Stodola-Vianello method (SVM) Present study Hertz (Hz)	Analytical solution Hannewald [8] Hertz (Hz)	Finite difference method (FDM) Hannewald [8] Hertz (Hz)			
			N=4	*N*=8	*N*=16	*N*=32
1	3.628659	3.629	3.568	3.610	3.623	3.627
2	7.445116	7.445	6.936	7.285	7.400	7.433
3	11.62200	11.622	9.794	11.031	11.454	11.577
4	16.306327	16.306	11.743	14.758	15.859	16.187
5	21.615579	21.616		18.29	20.632	21.350

			3		
6	27.639313	27.640	21.401	25.737	27.122
7	34.443624	34.444	23.841	31.091	33.522
8	42.076385	42.077	25.400	36.579	40.546

5. Discussion

The problems of torsional vibrations of mono-symmetric beams with closed cross-sections are governed by non-homogeneous variable parameter partial differential equation (PDEs). This is particularly where the vibration has external excitation and the elasticity and geometrical properties are variables due to non-homogeneous and non-prismatic cross-sections.

However, when the beam material is homogeneous and the cross-section is prismatic, and there is no excitation force, the problem becomes simplified to a homogeneous fourth order PDE with constant parameters. Such problems are solvable using techniques and methods for solving PDEs.

In this work, the PDE for the problem has been solved in closed form using Stodola-Vianello iteration method. The work assumed harmonic torsional vibrations and harmonic response of the torsional deflections. This assumption resulted in the expression of $\phi(x, t)$ in the form of a linear combination of sinusoidal functions depending only on t and unknown modal shape functions depending only on x.

Consequently, the independent spatial and time variables of the problem became de-coupled, and the resulting problem became a system of homogeneous ordinary differential equations of fourth order.

Stodola-Vianello iterations method (SVIM) was then applied via four successive integrations to obtain the SVIM iteration equations for the problem. The SVIM iteration equation contained four integration constants which were obtained using the four boundary conditions at the ends $x = 0$, and $x = l$ for the simply supported beam fully solved. The condition for convergence of the nth SVIM iteration was used to find the characteristic eigenvalue equation, which was then solved for the eigenvalues.

The natural frequencies were then determined for the first eight modes of torsional vibration and presented in Table 1, together with previous values obtained via analytical methods and FDMs by Hannewald [8]. Table 1 shows that the present SVIM natural frequencies are identical with the analytical results presented by Hannewald [8]. This illustrates that the SVIM results for f_n are exact. This is expected since the exact shape functions for simply supported beams were used in deriving the SVIM solutions.

6. Conclusions

This work has presented SVIM for solving the free torsional vibration problem of homogeneous, prismatic beams with closed cross-sections.

In conclusion

i. The SVIM iteration equation was obtained using four successive integrations and contains four integration constants.

ii. For simply supported ends, the exact shape function is a sinusoidal function and satisfies all the boundary conditions.

iii. The constants of integration are computed using the four boundary conditions.

iv. For simply supported ends, the characteristic eigenequation is an algebraic equation which is solved to find the eigenvalue ω_n from which f_n is calculated.

v. The expressions obtained for ω_n and f_n are exact and yield exact values for the frequencies for any given mode of torsional vibration.

vi. The effectiveness and accuracy of the SVIM has been demonstrated for free torsional vibration analysis of monosymmetric beam sections.

vii. The exact expression was used to sole the box beam bridge problem previously studied using analytical and FDM method and identical results were obtained in this work via SVIM.

NOMENCLATURE

x	longitudinal axial coordinate
y	coordinate in the breadth of the beam
z	transverse coordinate
t	time
$C_w(x)$	warping stiffness of the beam when there is variable warping stiffness for non-homogeneous beam material
$C(x)$	torsional stiffness that varies along the longitudinal axis of beam
ρ	mass density
I_p	polar moment of inertia
$\phi(x, t)$	angle of twist or torsional angle
$M_e(x, t)$	externally applied twisting moment or intensity of applied distributed torque
ρI_p	mass moment of inertia per unit length
I_{yy}	moment of inertia about yy axis
I_{zz}	moment of inertia about zz axis
$G(x)$	shear modulus that varies along the longitudinal axis
G	constant shear modulus for homogeneous beam material
$E(x)$	Young's modulus of elasticity for non-homogeneous beam material
E	Young's modulus of elasticity for homogeneous beam material
I_w	warping constant for homogeneous prismatic beam cross-sections
$I_w(x)$	warping constant that varies along the longitudinal axis
J	torsional moment of inertia or Saint Venant torsional constant
$J(x)$	torsional moment of inertia for non-homogeneous, non-prismatic beam cross sections
i	imaginary number in complex variables theory
$F_n(x)$	torsional vibration mode shape function for the nth mode
ω_n	angular frequency of torsional vibration for the nth mode
f_n	frequency of torsional vibration for the nth mode
$\frac{d^k}{dx^k}$	kth derivative with respect to x
l	length of beam
$\frac{\partial}{\partial x}$	operator for partial derivative with respect to x

$\frac{\partial^2}{\partial x^2}$ operator for second partial derivative with respect to x

$\frac{\partial^2}{\partial t^2}$ operator for second partial derivative with respect to t

c_1, c_2, c_3, c_4 constants of integration

$\int_0^x (\ )dx$ integration with respect to x

$\int_0^x \int_0^x (\ )dxdx$ two successive integrations with respect to x

Hz Hertz
m meter
n torsional vibration mode number
N number of finite difference grids used in the finite difference method

$\int_0^x \int_0^x \int_0^x \int_0^x (\ )dxdxdxdx$ four successive integrations with respect to x

GPDE governing partial differential equation
PDE partial differential equation
N Newton
kN kilo Newton
MN mega Newton
SVIM Stodola-Vianello iteration method
SVM Stodola-Vianello method
FDM finite differential method
kg kilogram
FEM finite element method
b width of beam
h depth of beam
b_i internal width (inner width)
b_e external width (outer width)
h_i internal depth (inner depth)
h_e external depth (outer depth)

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